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Vector Theory of Spontaneous Lorentz Violation

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Vector Theory of Spontaneous Lorentz Violation

Shu Hong Fung, Colby College

2006

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Abstract

Classical electromagnetism predicts two massless propagating modes, which are known as the two polarizations of the photon. On the other hand, if the Lorentz symmetry of classical electromagnetism is spontaneously broken, the new theory will still have two massless Nambu-Goldstone modes resembling the photon. If the Lorentz symmetry is broken by a bumblebee potential that allows for excitations out of the minimum, then massive modes arise. Furthermore, in curved spacetime, such massive modes will be created through a process other than the usual Higgs mechanism because of the dependence of the bumblebee potential on both the vector field and the metric tensor. Also, it is found that these massive modes do not propagate due to the extra constraints.

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1 Introduction

A Lorentz transformation is a linear transformation that preserves a space-time interval. In special relativity, Lorentz transformations consist of rotations and boosts; in general relativity, they are local transformations that rotate or boost a freely falling frame. A theory that is invariant under Lorentz transformation is said to be Lorentz symmetric. Lorentz symmetry plays an important role in particle physics, since it is the basis of relativity, which is in turn experimentally tested to extremely high degree of accuracy. Thus, any other physical theory is expected to be Lorentz symmetric so as to be compatible with relativity.

However, the general relativistic theory of gravity does not qualify as a quantum theory, in the sense that it is not normalizable. This loophole in general relativity motivates attempts of searching for a unified theory that would incorporate both general relativity and the standard model. Some well known candidates include supergravity, loop quantum gravity, as well as string theory.

Recent developments in the quantization of gravity suggest that Lorentz symmetry could be violated by a small amount [1]. Symmetry breaking could be categorized into two types, namely explicit and spontaneous symmetry breaking. We are interested in the latter possibility.

For a theory in which Lorentz symmetry is not violated, the Lagrangian of the theory, being a scalar function of some tensors, remains invariant when the tensors are transformed under Lorentz transformations. This invariance can be destroyed by introducing into the Lagrangian a potential term with specific properties. The first possibility is that the potential term does not obey the Lorentz invariance, in which case the symmetry is said to be violated explicitly. The second possibility, which in our case is of more significance, is that the potential term has non-unique vacua. If this is the case, symmetry would be broken once the physical vacuum is picked out from all possible ones. This leads to what is called spontaneous symmetry breaking (SSB).

The simplest example of a theory with spontaneous Lorentz breaking is one in which a vector field acquires a non-zero vacuum expectation value (vev). In some respects, such a theory would be similar to electromagnetism since it describes an interacting vector field resembling the 4-potential. However, in electromagnetism (without SSB) there is an unbroken $U(1)$ gauge symmetry. In this case, the only modes that appear are those that correspond to the massless gauge bosons. There are two massless propagating modes in total, which can be identified as the two polarizations of photons. Since the photon is massless, any proposed theory should always predict two massless modes in order to account for its presence.

Nonetheless, gauge symmetry is not necessary for massless modes to exist. Theories with spontaneous symmetry breaking also predict the existence of massless particles. As will be shown here, this result holds as well when it is Lorentz symmetry that is spontaneously broken. Hence it will be shown that it is possible to have a model with broken Lorentz symmetry without losing massless modes, which can therefore be considered as candidates for photons. The models that will be studied here are vector models resulted from modifying the electromagnetic Lagrangian.

In order to find out whether there is massless or massive modes, it is important to distinguish global and local symmetries. If a continuous symmetry is spontaneously broken, both massive and massless modes can appear. The massless modes, if there are any, are called Nambu-Goldstone (NG) modes. An NG mode is the result of the breaking of a global symmetry. The massive modes, if there are any, would indicate the occurrence of the Higgs mechanism. The breaking of a local symmetry is a necessary condition for the usual Higgs mechanism to occur. In the usual Higgs mechanism, the gauge fields acquire a mass. However, additional massive modes can arise as well due to the form of the potential inducing the symmetry breaking. Although NG modes and the usual Higgs mechanism for the case of spontaneous Lorentz violation have been extensively studied [2, 3, 4, 5], relatively little is known about these additional types of massive modes that might arise. This paper will concentrate on discussing the effects of these additional massive modes in the case of Lorentz violation.

In the next section, background on Lagrangian formulation and gauge symmetry will be given. These ideas and techniques will be used to formulate General Relativity (GR) in Section 3. In Section 4 we introduce the vector model, known as a bumblebee model, that creates spontaneous Lorentz violation. Sections 5 and 6 will examine the bumblebee model in flat and curved spacetime respectively. Section 7 will investigate the implications of the massive modes of the models. Finally, Section 8 will present the conclusion.

2 Background

We will investigate our models using the Lagrangian approach. A Lagrangian of a system is the difference between the kinetic energy and potential energy of the system. In the case of non-field theories, the terms in the Lagrangian are functions of position and velocity of the particles in the system. An example is the Lagrangian that describes the system of a charged particle in electric and magnetic fields, which is

$$L = \frac{1}{2} m \dot{\vec{x}}^2 - q\phi + \frac{q}{c} \dot{\vec{x}} \cdot \vec{A}, \tag{1}$$

where m is the mass of the particle, $\vec{x}$ the position vector, q the charge, ϕ the electric potential and $\vec{A}$ the magnetic vector potential. In the case of field theories, the terms in the Lagrangian are functions of various field tensors. An example would be the classical electromagnetic Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{2}$$

where $F^{\mu\nu}$ is the electromagnetic tensor. Note that $\mathcal{L}$ is used in place of L to represent the Lagrangian density, which is the Lagrangian per unit volume.

Nonetheless for brevity we will stick to the term Lagrangian, and whether it stands for density or not should be obvious from the context.

We would like to be able to extract other information from the Lagrangian. From the least action principle (or extreme action principle) we have

$$\delta S = \delta \int L \, \mathrm{d}t = \int \delta L \, \mathrm{d}t = 0, \tag{3}$$

which, after some manipulations, lead to the Euler-Lagrange Equation

$$\frac{\partial \mathcal{L}}{\partial \phi^\mu} - \frac{\partial}{\partial x^\nu} \frac{\partial \mathcal{L}}{\partial \phi^\mu_{,\nu}} = 0. \tag{4}$$

By using the Euler-Lagrange Equation we can get a set of differential equations, known as the equations of motion, from the Lagrangian of the theory. As an example, consider the classical electromagnetic Lagrangian in Eq.(2). The Lagrangian is a function of the electromagnetic field strength tensor $F^{\mu\nu}$, which in turns equals $\partial_\mu A_\nu - \partial_\nu A_\mu$ where A_μ is the four potential. Hence the Lagrangian is varied with respect to A_μ,

$$\begin{aligned} \delta L &= -\frac{1}{4} \delta (F^{\mu\nu} F_{\mu\nu}) \\ &= -(\partial_\mu \delta A_\nu)(\partial^\mu A^\nu) + (\partial_\mu \delta A_\nu)(\partial^\nu A^\mu) \\ &= \delta A_\nu (\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu). \end{aligned} \tag{5}$$

The third line comes from carrying out a partial integration and setting variations to zero at end points. Since the action should be at its extremum,

δL equals zero and hence

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = 0, \tag{6}$$

which give the equations of motion. Since ν can take the value of 0,1,2 or 3, this is in fact a set of four differential equations. The solution to this set of equations represents the modes that are predicted by the theory.

One important aspect of a theory is the set of symmetries that it possesses. If a theory has a certain symmetry, it would be invariant under the corresponding transformation. Also, from Noether's Theorem we are assured that there exists a one-to-one correspondence between a symmetry and a conserved quantity. Furthermore, it will be shown that the breaking of a symmetry could give rise to new propagating modes. Before the symmetries of our theory and their violations can be investigated, however, several concepts have to be established [6].

First of all, a symmetry can be either global or local. This difference is related to the nature of gauge choices. A gauge choice is the removal of redundant variables. To demonstrate the difference between global and local symmetries, consider a simple electric circuit that consists of a resistor and a $10V$ battery. Obviously, the potential at any point on the circuit can be defined to be any value, as long as the potential difference across the resistor is $10V$. In other words, the physics remain the same if we pick a new gauge by doing the transformation

$$V \to V + V_0. \tag{7}$$

Here, the gauge symmetry is global because when the potential is fixed at one point, the same value of V_0 will be determined for every other point. On the other hand, in some theories the symmetry transformations depend on the coordinates. In such cases the symmetries are said to be local. As an example, a theory is locally symmetric if it is invariant under the transformation

$$\phi \to e^{i\alpha(x)}\phi \tag{8}$$

where ϕ is some scalar field of the theory and α is a phase angle that is dependent on the one dimensional coordinate x.

A theory that has a local gauge symmetry always predicts the existence of massless modes. Such massless particles are called gauge bosons. These bosonic particles are carriers of fundamental forces. The photon, gluon and graviton are all examples of gauge bosons. Earlier when we discussed the equations of motion (6) that come from varying the Lagrangian of classical electromagnetism, it was pointed out that the solution to the equations of motion are the modes predicted by the theory. In this case, they represent the massless gauge boson of electromagnetism, namely the photon. We are about to verify this claim.

The set of equations of motion consists of four partial differential equations, hence one would naively expect the theory to predict four modes. Nonetheless, the study in electromagnetism shows that there are really only two physical modes, namely the two orthogonal polarizations of the photon. To reconcile the difference, it is important to spot the gauge redundancy in the theory. These redundant degrees of freedom, though resulting in good solutions to the equations of motion, represent non-physical modes that do not propagate. In this case, the Lagrangian is invariant under the transformation

$$A^\mu \to A^\mu + \partial^\mu \Lambda \tag{9}$$

where Λ is an arbitrary constant vector field. This accounts for one non-physical mode that can be transformed away. In addition, if in Eq.(6) we let $\nu = 0$, then

$$\begin{aligned} \Box A^0 - \partial^0 \partial_\mu A^\mu &= 0 \\ \partial_j \partial^j A^0 - \partial^0 \partial_j A^0 &= 0 \end{aligned} \tag{10}$$

where ∂_j denotes the spatial components of the differential operator. From the fact that terms in the form $\partial^0 \partial_0 A^0$ are absent we can conclude that the field component A_0 is an auxiliary mode that does not propagate and is unphysical. Furthermore, the gauge degree of freedom can be removed by observing from Eq.(9) that

$$\partial_\mu A^\mu \to \partial_\mu A^\mu + \partial_\mu \partial^\mu \Lambda, \tag{11}$$

which implies that if the constant field Λ is chosen so that $-\partial_\mu A^\mu = \partial_\mu \partial^\mu \Lambda$, the second term in Eq.(6) become zero, which effectively remove the gauge degree of freedom (this procedure is known as the Lorentz gauge). We are then left with

$$\Box A^\nu = 0, \tag{12}$$

with the two conditions

$$(1)\ A^0 \text{ is auxiliary}$$
$$\text{and } (2)\ \partial_\mu A^\mu = 0.$$

Hence the problem has been reduced to solving Eq.(12). At first sight this might seem tedious, but the theory of Fourier transforms suggests that we can require the solution to have sinusoidal form, since any other functions can be expressed as an integral of these base functions. Therefore, we take the solution to be

$$A^\mu = \varepsilon^\mu e^{-i\vec{K}\cdot\vec{x}} \tag{13}$$

where ε^μ is a constant 4-vector, $\vec{K}$ and $\vec{x}$ are energy-momentum and coordinate 4-vectors respectively. Even before solving the equations, we know in advance from condition (1) that $\varepsilon^0 = 0$, and from condition (2) that ε^μ and K^μ are perpendicular to each other, i.e. $K_\mu \varepsilon^\mu = 0$. Also, we have the freedom to pick a spatial direction so that $K^\mu = (K^0, 0, 0, K^3)$. Plugging A^μ into Eq.12, and using the two conditions, we find that the two propagating modes can be expressed as

$$A^\mu = \begin{cases} e^{-i\vec{K}\cdot\vec{x}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, & \text{for polarization in the x-direction} \\ e^{-i\vec{K}\cdot\vec{x}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, & \text{for polarization in the y-direction,} \end{cases} \tag{14}$$

which is consistent with the properties of the photon. In addition Eq.(12) in Fourier space yields the condition that $K^\mu K_\mu = 0$. This requires that $K^0 = K^3$ and is the condition that the propagating mode is massless.

This example is worked through in such detail because it demonstrates the procedures we will carry out in dealing with more complicated Lagrangians in later sections.

The second concept to be introduced, besides that of global and local symmetries, is that of symmetry breaking. Suppose that the Lagrangian describing a scalar field

$$\mathcal{L}_0 = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi \tag{15}$$

is invariant under some transformation Γ. Now, an additional potential can be added to the theory. If this extra potential term is *not* Γ-invariant, then the new Lagrangian $\mathcal{L} = \mathcal{L}_0 + V_e$ will not be Γ-symmetric. In this case, we say that the Γ-symmetry is violated *explicitly*. Explicit symmetry breaking of a theory does not create any new modes.

Explicit symmetry breaking is not the only possible way of violating a symmetry. Suppose the potential term that corresponds to the extra potential does obey Γ-invariance. To be specific, consider the case where both $\mathcal{L}_0$ and V_s, the potential term, are functions of the one-dimensional scalar field ϕ. If the potential is

$$V_s = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4$$

with $\lambda > 0$ and $m^2 < 0$, then the potential, being a function of ϕ, would have two local minima. These minima, also known as vacuum expectation values (vev), are denoted by

$$\langle\phi\rangle = \pm\sqrt{\frac{-m^2}{\lambda}}.$$

They signify the value of ϕ at which the system would attain the lowest possible energy. Suppose the Γ-transformation sends $\phi \to -\phi$. It is easy to verify that V_s is Γ-invariant. Hence one expects that Γ-symmetry is not broken by the introduction of V_s, except that in real world only one value of ϕ can be picked as the true vacuum at any moment. Once the true vacuum is picked, the field excitations about the vacuum would be redefined to be

$$\phi' = \phi - \langle\phi\rangle \tag{16}$$

which destroys the Γ-symmetry. In this case, we say that the Γ-symmetry is violated *spontaneously*.

The simple model of Γ-symmetry violation shown above is an example of discrete symmetry breaking, as there are only finitely many possible discrete vacuums (two in this case). We will now proceed to investigate the case of continuous symmetry breaking. In order for this to occur, the extra potential term V_s must depend on more than one field variable, hence we consider the next simplest case where V_t is a function of two scalar fields ϕ_1 and ϕ_2. Consider the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\,\partial_\mu \phi_1\,\partial^\mu \phi_1 + \frac{1}{2}\,\partial_\mu \phi_2\,\partial^\mu \phi_2 \tag{17}$$

and the potential

$$V_t = \frac{1}{2} m^2[(\phi_1)^2 + (\phi_2)^2] + \frac{1}{4}\lambda[(\phi_1)^2 + (\phi_2)^2]^2 \tag{18}$$

with $\lambda > 0$ and $m^2 < 0$. This potential, with its shape resembling a Mexican hat, has its minimum when

$$(\phi_1)^2 + (\phi_2)^2 = -\frac{m^2}{\lambda}. \tag{19}$$

Hence the vacuum of this model forms a continuous ring. Suppose a Θ-transformation is defined as the linear transformation

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Since the Θ-transformation is a rotation (in ϕ-space) by an angle θ, it is obvious that V_t is Θ-invariant. Note that Θ-symmetry is a continuous symmetry.

Once again, only one set of ϕ_1 and ϕ_2 can be the true physical vacuum. Once the true vacuum is picked, the Θ-symmetry will be violated spontaneously. As a footnote, both Γ-symmetry and Θ-symmetry are global symmetries.

It can be shown that the spontaneous breaking of a continuous global symmetry gives rise to massless modes known as Nambu-Goldstone modes [7]. We are going to demonstrate this using the Mexican hat potential described above, but the results will also apply to all cases of spontaneous breaking of global continuous symmetries. From Eq.(19) we can write

$$| \left\langle \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right\rangle | = \pm\sqrt{\frac{-m^2}{\lambda}} \equiv v \tag{20}$$

to denote the magnitude of the vev. Obviously, the ordered pair $(\phi_1\,, \phi_2) = (v, 0)$ is one of the possible vacua of the system. Suppose $(v, 0)$ is the physical vacuum that is picked. First of all it is necessary to redefine the fields, just as we did in Eq.(16), by

$$\begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} - \begin{pmatrix} v \\ 0 \end{pmatrix}. \tag{21}$$

Since we are interested in the behavior of the system being slightly excited about the vacuum, it is sufficient for us to consider the small excitations about the vacuum value. This is equivalent to linearizing the theory. Small excitation about the true vacuum are represented by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \eta \\ \xi \end{pmatrix} + \begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} \eta + v \\ \xi \end{pmatrix}, \tag{22}$$

where η and ξ are "small" quantities. Substituting this into Eq.(17) and dropping terms with order higher than quadratic, we obtain the Lagrangian to quadratic order

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 - V_t \\
&= \frac{1}{2}\,\partial_\mu\phi_1\,\partial^\mu\phi_1 + \frac{1}{2}\,\partial_\mu\phi_2\,\partial^\mu\phi_2 - \frac{1}{2}m^2[(\phi_1)^2 + (\phi_2)^2] - \frac{1}{4}\lambda[(\phi_1)^2 + (\phi_2)^2]^2 \\
&= \frac{1}{2}\,\partial_\mu\eta\,\partial^\mu\eta + \frac{1}{2}\,\partial_\mu\xi\,\partial^\mu\xi - \frac{1}{2}m^2[(v+\eta)^2 + (\xi)^2] - \frac{1}{4}\lambda[(v+\eta)^2 + (\xi)^2]^2 \\
&= \frac{1}{2}\,\partial_\mu\xi\,\partial^\mu\xi + \frac{1}{2}\,\partial_\mu\eta\,\partial^\mu\eta + \frac{m^2\eta^2}{2},
\end{aligned}
\tag{23}
$$

where we have used Eq.(20) in simplifying the expressions. The first term in the quadratic order Lagrangian can be identified as the kinetic term of the ξ mode, the second the kinetic term of the η mode, and finally the third the inertial term of the η mode. Hence in this case the spontaneous breaking of a continuous global symmetry results in two modes, one (η) massive and one (ξ) massless. The massless ξ mode is the NG mode. It represents an excitation that stays in the potentail minimum. The additional massive mode η is an excitation that does not stay in the minimum. It arises due to the shape of the potential V_t.

The next scenario to be considered is the spontaneous breaking of a continuous *local* symmetry. The following example will demonstrate how massive modes are created under this situation by the Higgs mechanism. As in the case of continuous global symmetry violation described above, we start with the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\,\partial_\mu \phi_1\,\partial^\mu \phi_1 + \frac{1}{2}\,\partial_\mu \phi_2\,\partial^\mu \phi_2, \tag{24}$$

which has dependence on both scalar fields ϕ_1 and ϕ_2. However, we now require the Θ-transformation to be a local transformation

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}\cos\theta(x) & \sin\theta(x)\\ -\sin\theta(x) & \cos\theta(x)\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{25}$$

Notice that the angle θ is now a function of position x. Since a rotation can be represented by the multiplication by a complex number, we may as well express Eq.(25) as

$$\phi \to e^{i\theta(x)T}\phi, \tag{26}$$

where T is the generator of the symmetry group. However, it does not take long for us to realize that the Lagrangian of Eq.(24) is *not* invariant under this local transformation. The reason is that partial derivatives are present in the kinetic term, and unlike the previous case where θ was a constant,

$$\begin{aligned}\partial_\mu\, e^{i\theta(x)T}\phi &= e^{i\theta(x)T}\partial_\mu\,\phi + \phi i T e^{i\theta(x)T}\partial_\mu\,\theta\\ &\neq e^{i\theta(x)T}\partial_\mu\,\phi.\end{aligned} \tag{27}$$

It is the presence of the extra term that causes the failure of maintaining local symmetry. To save the local symmetry, we introduce a new derivative

D_μ such that

$$D_\mu \, e^{i\theta(x)T}\phi \;\;=\;\; e^{i\theta(x)T} D_\mu \, \phi. \tag{28}$$

The use of the *gauge covariant derivative* D_μ requires us to introduce a new vector field A_μ in order to keep the derivatives covariant. Also, this gauge field has to transform under $e^{i\theta(x)T}$ appropriately so that Eq.(28) holds. After some manipulation it is found that the gauge covariant derivative is defined as

$$D_\mu = \partial_\mu + igA_\mu, \tag{29}$$

where g is the charge, and a gauge field that transforms as

$$A_\mu \to A_\mu - \frac{1}{g}\,\partial_\mu\theta \tag{30}$$

does the job. Corresponding to this gauge field, a new kinetic term appears in the Lagrangian that accounts for its propagation. And of course, this new term has to be invariant under the local transformation as well. To sum up, the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\,D_\mu\phi\,D^\mu\phi \tag{31}$$

is invariant under the local gauge transformation

$$\begin{aligned} \phi &\;\to\; e^{i\theta(x)T}\phi \\ A_\mu &\;\to\; A_\mu - \frac{1}{g}\,\partial_\mu\theta \end{aligned}$$

$$F_{\mu\nu} \quad \to \quad F_{\mu\nu}, \tag{32}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{33}$$

Now that the correct local symmetry is established (by the introduction of gauge vector field A_μ and covariant derivative D_μ), we can create local symmetry breaking by putting into the Lagrangian the potential term

$$V_t = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 \tag{34}$$

with $\lambda > 0$ and $m^2 < 0$ so that the complete Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\,D_\mu\phi\,D^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4. \tag{35}$$

To find the modes of this model with local symmetry, we once again consider the small excitations about the vacuum value. Expanding the Lagrangian in Eq.(35) to quadratic order gives

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g^2v^2}{2}A_\mu A^\mu + \frac{1}{2}\,\partial_\mu\varepsilon\,\partial^\mu\varepsilon + m^2\varepsilon^2 + \mathcal{I} \tag{36}$$

where $\mathcal{I}$ is a term that accounts for the interactions between A_μ and ε, where ε is the small excitation and v is the vev as given in Eq.(20). The reason of having only one parameter of small excitation, ε, instead of two in the case of global symmetry, ξ and η, is that we can always set the excitation in one direction to be zero by re-parameterizing the field ϕ.

Inspecting Eq.(36), we identify the first and second terms to be respectively the kinetic and inertial terms of the gauge vector field A_μ, and the third and forth terms to be respectively the kinetic and inertial terms of the scalar field ϕ. In other words, the breaking of this continuous local symmetry does not give us any massless mode. The expected massless mode

$$\frac{1}{2}\,\partial_\mu \xi \, \partial^\mu \xi, \tag{37}$$

where ξ is the excitation in the direction that we have gauged to zero through re-parameterization, has its degree of freedom being absorbed into the inertial term of the gauge field A_μ. This process, in which the introduction of a gauge field creates massive modes out of the degrees of freedom of massless modes, is known as the Higgs mechanism. Note, however, that there is an additional massive mode ε (the Higgs particle). In gauge theory, it is not a gauge field, but rather is an independent massive mode that arises due to the shape of the potential.

3 Lagrangian Formulation of General Relativity

So far, we have discussed the cases where symmetry is not broken (with the help of classical electromagnetism), where discrete symmetry is broken, where global continuous symmetry is broken and where local continuous symmetry is broken (all with the consideration of a scalar model). The ideas that are discussed will return throughout later sections. In this section, general relativity will be derived using a Lagrangian formulation [8].

The Lagrangian of general relativity is

$$\mathcal{L}_{GR} = \frac{1}{2\kappa}\sqrt{-g}\ R, \tag{38}$$

where $\kappa = 8\pi G$ (in units with $c = 1$) is a constant, g is the determinant of the metric tensor $g_{\mu\nu}$ and R is the Ricci scalar. Since the Ricci scalar can be obtained by applying a series of differential and algebraic operations on $g_{\mu\nu}$, $\mathcal{L}_{GR}$ is really nothing more than a function of the metric tensor. Hence it is the only quantity we will be varying $\mathcal{L}_{GR}$ with respect to. Varying Eq.(38) with respect to $g_{\mu\nu}$ gives us the equation of motion

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0, \tag{39}$$

where R and $R_{\mu\nu}$ obey the relation

$$R = R_{\mu}{}^{\mu}. \tag{40}$$

Eq.(39) is, not very surprisingly, Einstein's equation in vacuum. Furthermore, we assume that the curvature of spacetime is small, i.e. $g_{\mu\nu}$ deviates from $\eta_{\mu\nu}$, the special relativity metric, by only a small amount. As an aside, we follow the convention that $\eta_{\mu\nu} = diag(-1, +1, +1, +1)$.

While Eq.(39) is Einstein's equation in vacuum, the complete Einstein's equation would also contain a term that accounts for the presence of matter and energy density field. The complete Einstein's equation reads

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{41}$$

where G is the gravitational constant and $T_{\mu\nu}$ the stress-energy tensor. The form of $T_{\mu\nu}$ depends of course on the energy and matter distribution of the universe. For our purpose, we will confine our attention to Einstein's equation in vacuum in this section, and demonstrate how technology helps us in solving for the modes.

Solving for modes amounts to looking for expressions that solve Eq.(39). The main idea is the same as in the earlier discussion concerning classical electromagnetism, namely that we are justified in using sinusoidal waves as the eigen-modes. Nonetheless, general relativity is a matrix theory (of the four by four matrix $g_{\mu\nu}$), so there are totally 16 equations of motion, instead of 4 in classical electromagnetism, and hence 16 modes as a result. Fortunately, $g_{\mu\nu}$ is required to be symmetric, which reduces the number of independent solutions from 16 to 10. Still, this is a large number, and technology will prove to be useful in computing the eigen-modes.

We start by linearizing the equations of motion, with the assumption that the curvature of spacetime is small. If that is the case, then

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{42}$$

where $h^{\mu\nu}$ is a four-by-four symmetric matrix with small entries. Since $h^{\mu\nu}$ is small, anything higher than first order in this quantity is dropped. Hence the quantity $h^{\mu\nu}$ can have its indices being lowered (or raised) by contracting simply with $\eta_{\mu\nu}$ (instead of $g_{\mu\nu}$), yielding

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{43}$$

Notice the sign difference compared to Eq.(42). Substituting Eq.(42) and Eq.(43) into Eq.(39) and dropping terms with higher order than h^2, the Lagrangian to quadratic order is

$$\begin{aligned}\mathcal{L}_{GR} = \frac{1}{2}[(\partial_\mu h^{\mu\nu})(\partial_\nu h) - (\partial_\mu h^{\rho\sigma})(\partial_\rho h^\mu{}_\sigma) + \frac{1}{2}\,\eta^{\mu\nu}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) \\ - \frac{1}{2}\,\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu h)].\end{aligned} \tag{44}$$

Varying this Lagrangian with respect to $h^{\mu\nu}$ gives us the equations of motion

$$\frac{1}{2}\left(\partial_\sigma \partial_\nu h^\sigma{}_\mu + \partial_\sigma \partial_\mu h^\sigma{}_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho \partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu}\Box h\right) = 0. \tag{45}$$

To solve for the modes of Eq.(45) we suppose the eigen-modes to be of the form

$$h^{\mu\nu} = h^{\mu\nu} e^{-i\vec{K}\cdot\vec{x}}, \tag{46}$$

where the $h^{\mu\nu}$ on the right hand side represents constant polarization matrix. The Fourier transform turns Eq.(45) from a set of partial differential equations to non-differential simultaneous equations. Recall that there are totally 10 independent entries for $g_{\mu\nu}$. We re-label the 10 independent entries as

$$\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ * & g_{11} & g_{12} & g_{13} \\ * & * & g_{22} & g_{23} \\ * & * & * & g_{33} \end{pmatrix} = \begin{pmatrix} h_{(1)} & h_{(2)} & h_{(3)} & h_{(4)} \\ * & h_{(5)} & h_{(6)} & h_{(7)} \\ * & * & h_{(8)} & h_{(9)} \\ * & * & * & h_{(10)} \end{pmatrix}. \tag{47}$$

Using this notation, the Fourier transformed equations of motion, being a set of 10 equations with 10 unknowns, can be expressed as a 10 by 10 matrix. We saw from Eq.(45) that all of these equations have zeroes on the right hand side, therefore the desired eigen-modes can be solved for indirectly by finding the eigenvalues and eigenvectors of the 10 by 10 matrix, which can be readily accomplished with the assistance of appropriate computer software.

Before feeding the 10 by 10 matrix to the software for the solutions, conditions can be applied to simplify the matrix. One might be tempted to choose a particular gauge (e.g. harmonic gauge) and simplify the equations, but this is not necessary as it will be taken care of by the solver. On the other hand, we can simplify the calculation by picking a nicer frame of reference in the following way. Since Lorentz symmetry is not broken, any rotation will leave the Lagrangian invariant. Hence we have the freedom to rotate the coordinate system so that the 4-momentum vector has zero x- and y-components, by choosing an appropriate rotation matrix R such that

$$R\begin{pmatrix}K_0\\K_1\\K_2\\K_3\end{pmatrix}\rightarrow\begin{pmatrix}K_0\\0\\0\\K_3\end{pmatrix}.$$

After making the simplification, the 10 by 10 matrix becomes

$$M=\begin{pmatrix}
0 & 0 & 0 & 0 & -\frac{{K_3}^2}{2} & 0 & 0 & -\frac{{K_3}^2}{2} & 0 & 0\\
0 & \frac{{K_3}^2}{2} & 0 & 0 & 0 & 0 & -\frac{K_0K_3}{2} & 0 & 0 & 0\\
0 & 0 & \frac{{K_3}^2}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{K_0K_3}{2} & 0\\
0 & 0 & 0 & 0 & K_0K_3 & 0 & 0 & K_0K_3 & 0 & 0\\
-\frac{{K_3}^2}{2} & 0 & 0 & K_0K_3 & 0 & 0 & 0 & \frac{K^2}{2} & 0 & -\frac{{K_0}^2}{2}\\
0 & 0 & 0 & 0 & 0 & -\frac{K^2}{2} & 0 & 0 & 0 & 0\\
0 & -\frac{K_0K_3}{2} & 0 & 0 & 0 & 0 & \frac{{K_0}^2}{2} & 0 & 0 & 0\\
-\frac{{K_3}^2}{2} & 0 & 0 & K_0K_3 & \frac{K^2}{2} & 0 & 0 & 0 & 0 & -\frac{{K_0}^2}{2}\\
0 & 0 & -\frac{K_0K_3}{2} & 0 & 0 & 0 & 0 & 0 & \frac{{K_0}^2}{2} & 0\\
0 & 0 & 0 & 0 & -\frac{{K_0}^2}{2} & 0 & 0 & -\frac{{K_0}^2}{2} & 0 & 0
\end{pmatrix}, \tag{48}$$

where

$$\begin{aligned}
K^2 &= \begin{pmatrix}K_0\\0\\0\\K_3\end{pmatrix}\cdot\begin{pmatrix}K_0\\0\\0\\K_3\end{pmatrix}\\
&= \begin{pmatrix}K_0 & 0 & 0 & K_3\end{pmatrix}\begin{pmatrix}-1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}K_0\\0\\0\\K_3\end{pmatrix}\\
&= -{K_0}^2+{K_3}^2.
\end{aligned}$$

Notice that Matrix(48) is symmetric. The set of 10 Fourier transformed equations can be written as

$$M \begin{pmatrix} h_{(1)} \\ \vdots \\ h_{(10)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{49}$$

and so if we find the eigenvalues λ of M in terms of (K_0, K_1, K_2, K_3), Eq.(49) ensures that $\lambda = 0$, which gives us the properties of each eigen-mode. The eigen-modes themselves are of course given by the eigenvectors of M.

With the help of Mathematica the eigenvalues of M are found to be

$$\begin{aligned}\lambda \in \{0, 0, 0, 0, -\frac{K^2}{2}, -\frac{K^2}{2}, \frac{1}{2}({K_0}^2 + {K_3}^2), \frac{1}{2}({K_0}^2 + {K_3}^2),\\ \frac{1}{4}\left(-{K_0}^2 + {K_3}^2 - \sqrt{3}\sqrt{3{K_0}^4 + 10{K_0}^2{K_3}^2 + 3{K_3}^4}\right),\\ \frac{1}{4}\left(-{K_0}^2 + {K_3}^2 + \sqrt{3}\sqrt{3{K_0}^4 + 10{K_0}^2{K_3}^2 + 3{K_3}^4}\right)\}.\end{aligned} \tag{50}$$

Setting λ to be zero, the first to the fourth eigenvalues hold trivially, which implies that they are extra degrees of freedom that could have been gauged away. The fifth and sixth eigenvalues give us

$$-\frac{K^2}{2} = 0,$$

which are two massless propagating modes (Gauge bosons), and the final four eigenvalues, having $K^2 \neq 0$, turn out to be unphysical modes that lead to unphysical dispersion relations. The results given by this Lagrangian formulation of GR agree with what is found by the geometric approach [9]. The Mathematica matrix and solutions can be found in the Appendix.

4 Lorentz Violation and Bumblebee Model

We now introduce a model, known as a bumblebee model [3], that describes the physics when Lorentz symmetry is spontaneously broken. In this section we investigate the idea of Lorentz violation and discuss the characteristics of the potentials that could lead to such a symmetry breaking. In Section 5 we will assume that the curvature of spacetime equals zero, which corresponds to the absence of gravity, and study the modes of the model. The more general case where gravity is present will be considered in Section 6.

Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{51}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This Lagrangian is invariant under Lorentz transformations. A Lorentz transformation

$$X^\nu \rightarrow \Lambda^\nu{}_\mu X^\mu, \tag{52}$$

where X^ν is a 4-vector, is a transformation that preserves the spacetime intervals. In the case of flat spacetime, $\Lambda^\nu{}_\mu$ is the same at any coordinate point, and hence Lorentz symmetry is global. Notice that unlike in Section

2 where A_μ was introduced as a gauge field, here we are treating it as a propagating vector field in its own right. Now we add to this Lagrangian a potential term that depends on A_μ

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\alpha(A^\mu A_\mu + a^2)^2, \tag{53}$$

where α and a are undetermined constants. The potential term $\frac{1}{2}\alpha(A^\mu A_\mu + a^2)^2$ has effects on Eq.(51) similar to that of Eq.(19) on Eq.(17). Namely, the global Lorentz symmetry is spontaneously broken by this potential term.

It is easy to see that a preferred direction in spacetime is implied by this potential, because if the potential is set to be zero, then

$$A^\mu A_\mu = -a^2$$

and obviously the timelike 4-vector $A_\mu = (a, 0, 0, 0)$ would satisfy the requirement. Of course, just as in the case of the Mexican hat potential, there are infinitely many 4-vectors that satisfy the vacuum condition. Nonetheless for simplicity we will consider the physical vacuum to be at $A_\mu = (a, 0, 0, 0)$.

5 Bumblebee Model in Flat Spacetime

The introduction of the potential term $\frac{1}{2}\alpha(A^\mu A_\mu + a^2)^2$ causes the spontaneous breaking of Lorentz symmetry in the Lagrangian, as shown in Section 4. Here we carry on the analysis and investigate the consequences of the

SSB.

Varying the full Lagrangian in Eq.(53) with respect to A_μ, the equations of motion are

$$\Box A_\mu - \partial_\mu \partial^\nu A_\nu - 2\alpha A_\mu (A_\nu A^\nu + a^2) = 0. \tag{54}$$

Once again we are interested in the small excitation about the vacuum value, hence Eq.(54) is to be linearized. With

$$A_\mu = a_\mu + \varepsilon_\mu$$

where $a_\mu a^\mu = -a^2$, the linearized equations of motion are

$$\Box \varepsilon_\mu - \partial_\mu \partial^\nu \varepsilon_\nu - 4\alpha\, a_\mu\, a_\nu\, \varepsilon^\nu = 0. \tag{55}$$

Before proceeding to solving for the modes, it should be pointed out that by taking a partial derivative of Eq.(55) we get

$$a_\mu\, a_\nu\, \partial^\mu \varepsilon^\nu = 0 \tag{56}$$

as a constraint on our choices of the 4-vectors a_μ and ε_μ. Take the eigenmodes to be

$$\varepsilon_\mu = \epsilon_\mu e^{-i\vec{K}\cdot\vec{x}}, \tag{57}$$

where the ϵ_μ on the right hand side is a constant polarization vector. Plugging into Eq.(56), we find

$$\begin{aligned} a_\mu\, a_\nu\, \partial^\mu \varepsilon^\nu &= 0 \\ \Rightarrow a_\mu\, a_\nu\, K^\mu \varepsilon^\nu &= 0 \\ \Rightarrow a_\nu \varepsilon^\nu &= 0. \end{aligned} \tag{58}$$

In summary, with this constraint and the specific choice of physical vacuum we can analytically solve Eq.(55) by substituting

$$\begin{aligned} a_\mu &= (a, 0, 0, 0) \\ K_\mu &= (K_0, 0, 0, K_3) \\ \varepsilon_\mu &= (0, \epsilon_1, \epsilon_2, \epsilon_3)\, e^{-i\vec{K}\cdot\vec{x}}. \end{aligned} \tag{59}$$

On the other hand, the method that was used in the previous section can also be employed. We use Eq.(57) to turn Eq.(55) into a set of non-differential simultaneous equations, and solve them on a computer. The advantage is that we do not have to worry about any of the constraints. Since the Fourier transformed equations of motion are 4 equations with 4 unknowns $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$, we can use Mathematica to solve for the eigenvalues of the matrix

$$M = \begin{pmatrix} -4a^2\alpha + {K_3}^2 & 0 & 0 & -K_0K_3 \\ 0 & -K^2 & 0 & 0 \\ 0 & 0 & -K^2 & 0 \\ -K_0K_3 & 0 & 0 & {K_0}^2 \end{pmatrix}, \tag{60}$$

where

$$K^2 = -{K_0}^2 + {K_3}^2.$$

The eigenvalues λ of the matrix M are found to be

$$\lambda \in \{-K^2, -K^2, \frac{1}{2}\left(-4a^2\alpha + {K_0}^2 + {K_3}^2 - \sqrt{16\,a^2\alpha {K_0}^2 + (4a^2\alpha - {K_0}^2 - {K_3}^2)^2}\right),$$
$$\frac{1}{2}\left(-4a^2\alpha + {K_0}^2 + {K_3}^2 + \sqrt{16\,a^2\alpha {K_0}^2 + (4a^2\alpha - {K_0}^2 - {K_3}^2)^2}\right)\}. \quad (61)$$

Since we know that global Lorentz symmetry is violated, the two massless NG modes with $K^2 = 0$ are immediately identified. These two NG modes are massless transverse modes and they are candidates for the two polarizations of the photon in a universe with broken Lorentz symmetry. The two modes with $K^2 \neq 0$ cannot be propagating modes, as they result in field strength tensor $F_{\mu\nu}$ with all zero entries.

6 Bumblebee Model in Curved Spacetime

Lorentz transformations were introduced in Section 4 as global gauge transformations in flat spacetime. In other words, a single ${\Lambda_a}^b$ can be applied to all 4-vectors regardless of their positions in spacetime. In Section 5 it was shown that a model with the breaking of this global symmetry produces massless modes as well as (unphysical) massive modes. Now suppose that in a more general setting, the curvature of the spacetime is not zero. This is equivalent to the presence of gravity according to Einstein's general relativity. If that is the case, the spacetime is not Euclidean anymore. In this case, no single ${\Lambda_a}^b$ can preserve the spacetime interval at all positions, hence the best we can have is a local Lorentz symmetry. Another way of expressing this is that unlike in the flat spacetime scenario where an inertial frame is global, the curvature of spacetime demands that any inertial frame be local.

Since the curvature is now non-zero, a new term has to be added to the Lagrangian in Eq.(53) in order to account for it. The Lagrangian of a bumblebee model in curved spacetime is therefore

$$\mathcal{L} = \sqrt{-g}\left[\frac{1}{2\kappa}R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\alpha(A^\mu A_\mu + a^2)^2\right], \tag{62}$$

where κ is a constant, g is the determinant of the metric tensor $g_{\mu\nu}$ and R is the Ricci scalar. Similar to the flat spacetime case, a preferred direction is implied by the potential term $-\frac{1}{2}\alpha(A^\mu A_\mu + a^2)^2$. One important difference from the flat spacetime case is that in curved spacetime this potential term also implicitly depends on $g_{\mu\nu}$, the metric tensor that describes the curvature. We can make it more obvious by re-writing the potential term as

$$-\frac{1}{2}\alpha(A^\mu A_\mu + a^2)^2 = -\frac{1}{2}\alpha(A_\nu g^{\mu\nu} A_\mu + a^2)^2.$$

As $g_{\mu\nu}$ (the gauge field) appears in the potential, it leads to consequences that differ from the usual Higgs mechanism (where V is a function of the field ϕ but not the gauge field). Hence the form of this potential term potentially provide a new way for massive modes involving the metric $g_{\mu\nu}$ to appear in the theory.

Because of the fact that the Lagrangian of Eq.(62) depends on both A_μ and $g_{\mu\nu}$, the Lagrangian needs to be varied with respect to both of them. Thus we expect two sets, instead of one, of equations of motion. The A_μ-equations are

$$\Box A^\nu - \partial_\mu \partial^\nu A^\mu - 2\alpha A^\nu (A_\mu A^\mu + a^2) = 0, \tag{63}$$

while the $g_{\mu\nu}$-equations are

$$\begin{aligned} -\frac{1}{2} F_{\mu\sigma} g^{\sigma\rho} F_{\nu\rho} - \alpha(A_\sigma A^\sigma + a^2) A_\mu A_\nu + \frac{1}{8} F_{\rho\sigma} F^{\rho\sigma} g_{\mu\nu} \\ + \frac{1}{4}\alpha(A_\sigma A^\sigma + a^2)^2 g_{\mu\nu} + R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0, \end{aligned} \tag{64}$$

where

$$R = R^\mu{}_\mu.$$

Since we did not expand the Lagrangian to quadratic order before doing the variation, the two sets of equations of motion should now be linearized. Now that we are working in a curved spacetime, $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$ is used to raise and lower indices. We must be careful in defining a consistent set of linearized components. We start by defining the vector A_μ to be

$$A_\mu = a_\mu + \varepsilon_\mu,$$

thus confining it to be a small excitation ε_μ about the vacuum value. The tensor $g_{\mu\nu}$ is defined to be

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

which suggests that the geometry of the spacetime deviates from an Euclidean one by only a small quantity $h_{\mu\nu}$. The corresponding metric with upper indices that is consistent with $g_{\mu\nu}$ would then be

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}.$$

The corresponding vector A^μ is not simply $a^\mu + \varepsilon^\mu$, but has to be found by using $g^{\mu\nu}$ and A_μ so that

$$\begin{aligned} A^\mu &= g^{\mu\nu} A_\nu \\ &= (\eta^{\mu\nu} - h^{\mu\nu})(a_\nu + \varepsilon_\nu) \\ &= a^\mu + \varepsilon^\mu - h^{\mu\nu} a_\nu, \end{aligned} \tag{65}$$

where the term $h^{\mu\nu}\varepsilon_\nu$ is dropped because of the assumption that both $h^{\mu\nu}$ and ε_ν are infinitesimal. Notice the presence of the extra term $-h^{\mu\nu}a_\nu$ that results from the coupling of A_μ and $g_{\mu\nu}$.

Substituting these expressions of A_μ, A^μ, $g_{\mu\nu}$ and $g^{\mu\nu}$ into Eq.(63) and

Eq.(64), the linearized equations of motion are

$$\Box\varepsilon^{\nu} - \partial_{\mu}\partial^{\nu}\varepsilon^{\mu} - 2\alpha\left(2a_{\sigma}\varepsilon^{\sigma} - h^{\sigma\rho}a_{\sigma}a_{\rho}\right)a^{\nu} = 0 \tag{66}$$

and

$$\begin{aligned} &- 2\alpha\left(2a_{\sigma}\varepsilon^{\sigma} - h^{\sigma\rho}a_{\sigma}a_{\rho}\right)a_{\mu}a_{\nu} + \partial_{\sigma}\partial_{\nu}h^{\sigma}{}_{\mu} \\ &\qquad + \partial_{\sigma}\partial_{\mu}h^{\sigma}{}_{\nu} - \partial_{\mu}\partial_{\nu}h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_{\sigma}\partial_{\rho}h^{\sigma\rho} + \eta_{\mu\nu}\Box h = 0. \end{aligned} \tag{67}$$

These two sets of equations are invariant under active diffeomorphism, namely the set of transformations

$$\begin{aligned} h^{\mu\nu} &\rightarrow h^{\mu\nu} - \partial^{\mu}\xi^{\nu} - \partial^{\nu}\xi^{\mu} \\ \varepsilon^{\mu} &\rightarrow \varepsilon^{\mu} - (\partial^{\mu}\xi^{\sigma})a_{\sigma}. \end{aligned}$$

The total number of equations of motion is 14. The solutions of 10 of them (the $g_{\mu\nu}$-equations) will give the 10 independent entries of $h_{\mu\nu}$ and the solutions of the other four (the A_{μ} equations) will give the four components of ε_{μ}. Before carrying out the Fourier transformation, we relabel the 14 variables to be

$$
\begin{pmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ * & h_{11} & h_{12} & h_{13} \\ * & * & h_{22} & h_{23} \\ * & * & * & h_{33} \end{pmatrix} = \begin{pmatrix} h_{(1)} & h_{(2)} & h_{(3)} & h_{(4)} \\ * & h_{(5)} & h_{(6)} & h_{(7)} \\ * & * & h_{(8)} & h_{(9)} \\ * & * & * & h_{(10)} \end{pmatrix}
$$

$$
\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix} = \begin{pmatrix} h_{(11)} \\ h_{(12)} \\ h_{(13)} \\ h_{(14)} \end{pmatrix} \tag{68}
$$

so that we can treat all the variables as a column vector with 14 components $h_{(1)}$ to $h_{(14)}$. The eigen-modes are taken to be

$$
h_{(k)} = h_{(k)} e^{-i\vec{K}\cdot\vec{x}}, \tag{69}
$$

where k ranges from 1 to 14.

The Fourier transformed equations of motion are a set of 14 non-differential simultaneous equations with 14 unknowns $h_{(1)}$ to $h_{(14)}$. They can be arranged to form a 14 by 14 matrix. Once again, we require that

$$
\begin{aligned} a_\mu &= (a, 0, 0, 0) \\ K_\mu &= (K_0, 0, 0, K_3). \end{aligned} \tag{70}
$$

The computation by Mathematica returns 14 eigenvalues. Setting the eigenvalues to be zero, four of them hold trivially, and hence are gauge abundances; four requires that $K^2 = 0$ (massless modes) and six requires that $K^2 \neq 0$.

7 Massive Modes

We have been suspending judgements on the implication of $K^2 \neq 0$ modes in both Sections 5 and 6. In Section 6 we computed the modes in a bumblebee model where the continuous local Lorentz symmetry is violated spontaneously. Our experience from Section 2 suggests that physical massive modes should be produced in this process because of the Higgs mechanism. And in fact we found in the end of the previous section that six modes with $K^2 \neq 0$ are predicted. However, though the approach using Mathematica is good for demonstrating the existence of the massless modes, it is inconclusive in determining whether the terms with $K^2 \neq 0$ are physical massive modes or unphysical auxiliary modes. The question to ask is therefore whether these six candidates are really propagating massive modes. We will do an analytic analysis of the equation of motion to answer this.

It would be instructive to start the discussion with the bumblebee model in flat spacetime. It was claimed in the end of Section 5 that the $K^2 \neq 0$ modes do not propagate. To see why this is the case, recall the constraint we have found in Eq.(58) that

$$a_\mu a_\nu \partial^\mu \varepsilon^\nu = 0.$$

In the case where $a_\mu = (a, 0, 0, 0)$, this constraint becomes

$$a^2 K^0 \varepsilon^0 = 0.$$

This relation has to hold in addition to the four equations of motion. Now suppose that $K_1 = K_2 = 0$, as it has been assumed in all timelike cases in previous sections. The first possibility is that

$$\begin{cases} K_0 \neq 0 \\ K_3 \neq 0. \end{cases} \tag{71}$$

In this case, it can be shown that no solution with $K^2 \neq 0$ can be found. In the second possibility where

$$\begin{cases} K_0 \neq 0 \\ K_3 = 0, \end{cases} \tag{72}$$

the only way to satisfy all five equations is to have $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$, which is clearly unphysical. Finally there is the possibility where

$$\begin{cases} K_0 = 0 \\ K_3 \neq 0. \end{cases} \tag{73}$$

In this case, either ε^0 or ε^3 is the only non-zero component. No matter which one the non-zero component is, K^2 would end up having the wrong sign, which forces the total energy E to be zero. It is for this reason that there are no propagating modes with $K^2 \neq 0$.

The situation in curved spacetime is more complicated. It was shown by Kostelecký and Samuel that the usual Higgs mechanism (in which $h_{\mu\nu}$ acquires a mass by absorbing an NG mode) does not occur [1], hence other possibilities must be explored. It turns out that for the bumblebee model in curved spacetime, the key to the presence of massive modes is the form of the potential term. In the scalar gauge theory, the local symmetry is spontaneously broken by the potential term V_t in Eq.(18), V_t depending only on the scalar field ϕ and not the gauge vector field A_μ. On the other hand, in the bumblebee model in curved spacetime, the potential term depends on both A_μ and $g_{\mu\nu}$. It is this difference that leads to the possibility of there being additional modes. The vector field ε_μ is the (non-gauge) field, while $h_{\mu\nu}$ is the gauge field of the gravitational interaction.

The analysis of the curved spacetime case is similar to what is done in flat spacetime. To recapitulate, the equations of motion are

$$\Box\varepsilon^\nu - \partial_\mu\partial^\nu\varepsilon^\mu - 2\alpha\left(2a_\sigma\varepsilon^\sigma - h^{\sigma\rho}a_\sigma a_\rho\right)a^\nu = 0 \tag{74}$$

and

$$\begin{aligned} &- 2\alpha\left(2a_\sigma\varepsilon^\sigma - h^{\sigma\rho}a_\sigma a_\rho\right)a_\mu a_\nu + \partial_\sigma\partial_\nu h^\sigma{}_\mu \\ &\qquad + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\sigma\partial_\rho h^{\sigma\rho} + \eta_{\mu\nu}\Box h = 0. \end{aligned} \tag{75}$$

Since A_μ and $g_{\mu\nu}$ are coupled, we define a new quantity

$$\beta = a^\mu \left(\varepsilon_\mu - \frac{1}{2} h_{\mu\nu} a^\nu \right). \tag{76}$$

Looking at Eq.(74) and Eq.(75), we can readily identify β as the term in parentheses. Also, it is obvious from the comparison with the $U(1)$ gauge model that β is responsible for the extra massive modes. Taking partial on Eq.(74) gives us the constraint

$$a^\mu \partial_\mu \beta = 0 \tag{77}$$

on the massive modes. If β is taken to be a plane wave $\beta = \beta e^{-i\overrightarrow{K}\cdot\overrightarrow{x}}$, then the constraint would become $a^\mu K_\mu = 0$.

Choosing the right gauge can largely simplify the calculation. In investigating propagation, the most convenient gauge choice is the harmonic gauge which requires that

$$\partial_\mu \bar{h}^{\mu\nu} = \partial_\mu \left(h^{\mu\nu} - \frac{1}{2}\, \eta_{\mu\nu} h \right) = 0. \tag{78}$$

Now we multiply Eq.(74) by a_ν and Eq.(75) by $-\frac{1}{2} a^\mu a^\nu$. Equating the left hand sides of the two (both equal zero) and applying the harmonic gauge, we have

$$\Box \beta - 4\alpha a^\nu a_\nu \beta - 2\alpha\, a_\mu a^\mu a_\nu a^\nu \beta = a_\nu \partial_\sigma \partial^\nu \varepsilon^\sigma, \tag{79}$$

which can be interpreted as the propagation of the coupled quantity β. This justifies our letting β to be a plane wave. Substituting $\beta = \beta e^{-i\vec{K}\cdot\vec{x}}$ into Eq.(79) gives

$$-K^{\mu}K_{\mu} - 4\alpha a^{\nu}a_{\nu} - 2\alpha\, a_{\mu}a^{\mu}a_{\nu}a^{\nu} = 0. \tag{80}$$

The right hand side is zero because of the $a^{\mu}K_{\mu} = 0$ constraint. This dispersion relation shows that a mass parameter can be defined for β,

$$-K^{\mu}K_{\mu} = M_{\beta}^{2} = 4\alpha a^{\nu}a_{\nu} + 2\alpha\, a_{\mu}a^{\mu}a_{\nu}a^{\nu}. \tag{81}$$

In the case where the vacuum value is time like, we can always set $a_{\mu} = (a, 0, 0, 0)$. Then

$$M_{\beta}^{2} = -4\alpha a^{2} + 2\alpha a^{4}. \tag{82}$$

But the $a^{\mu}K_{\mu} = 0$ constraint forces the total energy K_0 to be zero, and so the magnitude of momentum obeys $|\vec{p}| = |M_{\beta}|$, where $|M_{\beta}|$ is fixed as shown in Eq.(82). Such a mode with no time dependence and fixed spatial magnitude cannot form a physical wave packet. Hence we conclude that this mode, although having a well defined mass parameter, cannot propagate because the $a^{\mu}K_{\mu} = 0$ constraint forces the total energy to be zero.

Although the massive modes fail to propagate, their effects are manifested in the static limit. In the static limit, we consider the scenario where a small, constant massive charge is located at the origin to act as a source of electromagnetic and gravitational forces. The new Lagrangian is

$$\mathcal{L} = \sqrt{-g}\left[\frac{1}{2\kappa}R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\alpha(A^{\mu}A_{\mu} + a^2)^2 + A_{\mu}J^{\mu}\right], \qquad (83)$$

where J_{μ}, the 4-current, accounts for the charge of the source. Its mass is contained in the energy-momentum tensor $T_{\mu\nu}$, which is connected to $R_{\mu\nu}$ through the relation

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right).$$

It can be shown by making approximations on Eq.(83) that the massive modes of the bumblebee models moderate the Newtonian gravitational potential and the Coulomb potential. The seeking of the exact forms of moderated potentials is however beyond the scope of this paper.

8 Summary and Conclusion

The concepts of Lagrangian and gauge symmetry were introduced. A gauge symmetry can be global or local, and the consequences of symmetry breaking are different for the two. In a theory with unbroken local $U(1)$ gauge symmetry, massless gauge bosons were predicted. A symmetry can be violated explicitly, but the more interesting case is when it is violated spontaneously if a physical vacuum is picked among more than one mathematically equivalent vacua.

A scalar gauge model was investigated, with its symmetry being broken by a Mexican hat potential. If the symmetry is global, then the excitation about the vacuum value can have both massless and massive modes. Such massless modes are called Nambu-Goldstone (NG) modes. On the other hand, if the symmetry is local, a gauge vector field together with a gauge covariant derivative are required so that the fields transform correctly. The introduction of gauge field results in an extra kinetic term in the Lagrangian, which absorbs one degree of freedom from the field. This process, known as the Higgs mechanism, creates an extra massive mode.

This Lagrangian method was used to formulate general relativity in vacuum. The Lagrangian gives the correct Einstein's equation in vacuum as well as the two massless propagating modes.

All special relativistic theories are Lorentz symmetric. In other words, the spacetime interval is invariant under Lorentz transformation. We showed that the Lorentz invariance of a theory is spontaneously broken by the introduction of a bumblebee potential $-\frac{1}{2}\alpha(A^\mu A_\mu + a^2)^2$. In flat spacetime, the bumblebee model still predicts two massless modes, which can possibly be identified as the photon. Similarly, four massless modes are predicted by the bumblebee model in curved spacetime.

The modes with $K^2 \neq 0$ in the bumblebee models require more care. In the flat spacetime case, it is the extra constraint, in addition to the equations of motion, that prevents any massive mode to propagate. In the curved spacetime case, the conventional type of Higgs mechanism is not possible;

however, due to the dependence of the bumblebee potential on both A_μ and $g_{\mu\nu}$ an alternative mechanism becomes possible. In this alternative approach, the coupling between A_μ and $g_{\mu\nu}$ could lead to other mechanisms that create massive modes. Our analysis showed that such modes do not propagate, once again because of the extra constraint.

There are clearly a number of additional questions that could be addressed. These include looking for phenomenological tests that could distinguish between photon described using $U(1)$ gauge theory versus those arising as NG modes when Lorentz symmetry is spontaneously broken. In addition, although we have shown that the massive modes do not propagate as physical modes, their effects on Newtonian gravity, cosmology and general relativity remain open issues for future work.

9 References

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Appendix

Note: $(K_a\ \ K_b\ \ K_c\ \ K_d)$ in this Appendix is equivalent to $(K_0\ \ K_1\ \ K_2\ \ K_3)$ in the paper.

Lagrangian Formulation of General Relativity:

$$
GR = \begin{pmatrix}
0 & 0 & 0 & 0 & -\frac{K_d^2}{2} & 0 & 0 & -\frac{K_d^2}{2} & 0 & 0 \\
0 & \frac{K_d^2}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} K_a K_d & 0 & 0 & 0 \\
0 & 0 & \frac{K_d^2}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} K_a K_d & 0 \\
0 & 0 & 0 & 0 & K_a K_d & 0 & 0 & K_a K_d & 0 & 0 \\
-\frac{K_d^2}{2} & 0 & 0 & K_a K_d & 0 & 0 & 0 & \frac{1}{2}(-K_a^2 + K_d^2) & 0 & -\frac{K_a^2}{2} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}(K_a^2 - K_d^2) & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} K_a K_d & 0 & 0 & 0 & 0 & \frac{K_a^2}{2} & 0 & 0 & 0 \\
-\frac{K_d^2}{2} & 0 & 0 & K_a K_d & \frac{1}{2}(-K_a^2 + K_d^2) & 0 & 0 & 0 & 0 & -\frac{K_a^2}{2} \\
0 & 0 & -\frac{1}{2} K_a K_d & 0 & 0 & 0 & 0 & 0 & \frac{K_a^2}{2} & 0 \\
0 & 0 & 0 & 0 & -\frac{K_a^2}{2} & 0 & 0 & -\frac{K_a^2}{2} & 0 & 0
\end{pmatrix}
$$

$$
\text{Eigenvalues}[GR] = \left\{0, 0, 0, 0, \frac{1}{2}\left(K_a^2 - K_d^2\right), \frac{1}{2}\left(K_a^2 - K_d^2\right), \frac{1}{2}\left(K_a^2 + K_d^2\right), \frac{1}{2}\left(K_a^2 + K_d^2\right), \frac{1}{4}\left(-K_a^2 + K_d^2 - \sqrt{3}\sqrt{3K_a^4 + 10K_a^2K_d^2 + 3K_d^4}\right), \frac{1}{4}\left(-K_a^2 + K_d^2 + \sqrt{3}\sqrt{3K_a^4 + 10K_a^2K_d^2 + 3K_d^4}\right)\right\}
$$

Bumblebee Model in Flat Spacetime:

$$
Flat = \begin{pmatrix}
-4a^2\alpha + K_d^2 & 0 & 0 & -K_a K_d \\
0 & K_a^2 - K_d^2 & 0 & 0 \\
0 & 0 & K_a^2 - K_d^2 & 0 \\
-K_a K_d & 0 & 0 & K_a^2
\end{pmatrix}
$$

$$
\text{Eigenvalues}[Flat] = \left\{K_a^2 - K_d^2, K_a^2 - K_d^2, \frac{1}{2}\left(-4a^2\alpha + K_a^2 + K_d^2 - \sqrt{16a^2\alpha K_a^2 + (4a^2\alpha - K_a^2 - K_d^2)^2}\right), \frac{1}{2}\left(-4a^2\alpha + K_a^2 + K_d^2 + \sqrt{16a^2\alpha K_a^2 + (4a^2\alpha - K_a^2 - K_d^2)^2}\right)\right\}
$$

$$
\text{Eigenvectors}[Flat] = \left\{\{0, 0, 1, 0\}, \{0, 1, 0, 0\}, \left\{-\frac{-4a^2\alpha - K_a^2 + K_d^2 - \sqrt{16a^2\alpha K_a^2 + (4a^2\alpha - K_a^2 - K_d^2)^2}}{2K_aK_d}, 0, 0, 1\right\}, \left\{-\frac{-4a^2\alpha - K_a^2 + K_d^2 + \sqrt{16a^2\alpha K_a^2 + (4a^2\alpha - K_a^2 - K_d^2)^2}}{2K_aK_d}, 0, 0, 1\right\}\right\}
$$

Bumblebee Model in Curved Spacetime:

$$\text{Curved} = \begin{pmatrix}
-a^4 \alpha & 0 & 0 & 0 & -\frac{1}{2} K_d^2 & 0 & 0 & -\frac{1}{2} K_d^2 & 0 & 0 & -2 a^3 \alpha & 0 & 0 & 0 \\
0 & K_d^2 & 0 & 0 & 0 & 0 & -K_a K_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & K_d^2 & 0 & 0 & 0 & 0 & 0 & -K_a K_d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & K_a K_d & 0 & 0 & K_a K_d & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} K_d^2 & 0 & 0 & K_a K_d & 0 & 0 & 0 & \frac{1}{2} (-K_a^2 + K_d^2) & 0 & -\frac{1}{2} K_a^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & K_a^2 - K_d^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -K_a K_d & 0 & 0 & 0 & 0 & K_a^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} K_d^2 & 0 & 0 & K_a K_d & \frac{1}{2} (-K_a^2 + K_d^2) & 0 & 0 & 0 & 0 & -\frac{1}{2} K_a^2 & 0 & 0 & 0 & 0 \\
0 & 0 & -K_a K_d & 0 & 0 & 0 & 0 & 0 & K_a^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} K_a^2 & 0 & 0 & -\frac{1}{2} K_a^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 a^3 \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 a^2 \alpha + K_d^2 & 0 & 0 & -K_a K_d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_a^2 - K_d^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_a^2 - K_d^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -K_a K_d & 0 & 0 & K_a^2
\end{pmatrix}$$

$$\text{Eigenvalues[Curved]} = \left\{0, 0, 0, 0, \frac{1}{2} \left(K_a^2 - K_d^2\right), K_a^2 - K_d^2, K_a^2 - K_d^2, K_a^2 - K_d^2, K_a^2 + K_d^2, K_a^2 + K_d^2, f_1 (K), f_2 (K), f_3 (K), f_4 (K)\right\}$$

where $f_1 (K)$, $f_2 (K)$, $f_3 (K)$,
$f_4 (K)$ are functions of K such that $K^2 \neq 0$.