Appendix

Note: (K_a K_b K_c K_d) in this Appendix is equivalent to (K_0 K_1 K_2 K_3) in the paper.

Lagrangian Formulation of General Relativity:

$$GR = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{K_d^2}{2} & 0 & 0 & -\frac{K_d^2}{2} & 0 & 0 \\ 0 & \frac{K_d^2}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} K_a K_d & 0 & 0 & 0 \\ 0 & 0 & \frac{K_d^2}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} K_a K_d & 0 \\ 0 & 0 & 0 & 0 & K_a K_d & 0 & 0 & K_a K_d & 0 & 0 \\ -\frac{K_d^2}{2} & 0 & 0 & K_a K_d & 0 & 0 & 0 & \frac{1}{2}(-K_a^2 + K_d^2) & 0 & -\frac{K_a^2}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(K_a^2 - K_d^2) & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} K_a K_d & 0 & 0 & 0 & 0 & \frac{K_a^2}{2} & 0 & 0 & 0 \\ -\frac{K_d^2}{2} & 0 & 0 & K_a K_d & \frac{1}{2}(-K_a^2 + K_d^2) & 0 & 0 & 0 & 0 & -\frac{K_a^2}{2} \\ 0 & 0 & -\frac{1}{2} K_a K_d & 0 & 0 & 0 & 0 & 0 & \frac{K_a^2}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{K_a^2}{2} & 0 & 0 & -\frac{K_a^2}{2} & 0 & 0 \end{pmatrix}$$

$$\text{Eigenvalues[GR]} = \left\{0, 0, 0, 0, \frac{1}{2}\left(K_a^2 - K_d^2\right), \frac{1}{2}\left(K_a^2 - K_d^2\right), \frac{1}{2}\left(K_a^2 + K_d^2\right), \frac{1}{2}\left(K_a^2 + K_d^2\right), \frac{1}{4}\left(-K_a^2 + K_d^2 - \sqrt{3}\sqrt{3 K_a^4 + 10 K_a^2 K_d^2 + 3 K_d^4}\right), \frac{1}{4}\left(-K_a^2 + K_d^2 + \sqrt{3}\sqrt{3 K_a^4 + 10 K_a^2 K_d^2 + 3 K_d^4}\right)\right\}$$

Bumblebee Model in Flat Spacetime:

$$Flat = \begin{pmatrix} -4 a^2 \alpha + K_d^2 & 0 & 0 & -K_a K_d \\ 0 & K_a^2 - K_d^2 & 0 & 0 \\ 0 & 0 & K_a^2 - K_d^2 & 0 \\ -K_a K_d & 0 & 0 & K_a^2 \end{pmatrix}$$

$$\text{Eigenvalues[Flat]} = \left\{K_a^2 - K_d^2, K_a^2 - K_d^2, \frac{1}{2}\left(-4 a^2 \alpha + K_a^2 + K_d^2 - \sqrt{16 a^2 \alpha K_a^2 + (4 a^2 \alpha - K_a^2 - K_d^2)^2}\right), \frac{1}{2}\left(-4 a^2 \alpha + K_a^2 + K_d^2 + \sqrt{16 a^2 \alpha K_a^2 + (4 a^2 \alpha - K_a^2 - K_d^2)^2}\right)\right\}$$

$$\text{Eigenvectors[Flat]} = \left\{\{0, 0, 1, 0\}, \{0, 1, 0, 0\}, \left\{-\frac{-4 a^2 \alpha - K_a^2 + K_d^2 - \sqrt{16 a^2 \alpha K_a^2 + (4 a^2 \alpha - K_a^2 - K_d^2)^2}}{2 K_a K_d}, 0, 0, 1\right\}, \left\{-\frac{-4 a^2 \alpha - K_a^2 + K_d^2 + \sqrt{16 a^2 \alpha K_a^2 + (4 a^2 \alpha - K_a^2 - K_d^2)^2}}{2 K_a K_d}, 0, 0, 1\right\}\right\}$$

Bumblebee Model in Curved Spacetime:

$$\text{Curved} = \begin{pmatrix}
-a^4\,\alpha & 0 & 0 & 0 & -\frac{1}{2}K_d^2 & 0 & 0 & -\frac{1}{2}K_d^2 & 0 & 0 & -2\,a^3\,\alpha & 0 & 0 & 0 \\
0 & K_d^2 & 0 & 0 & 0 & 0 & -K_a K_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & K_d^2 & 0 & 0 & 0 & 0 & 0 & -K_a K_d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & K_a K_d & 0 & 0 & K_a K_d & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2}K_d^2 & 0 & 0 & K_a K_d & 0 & 0 & 0 & \frac{1}{2}(-K_a^2+K_d^2) & 0 & -\frac{1}{2}K_a^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & K_a^2-K_d^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -K_a K_d & 0 & 0 & 0 & 0 & K_a^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2}K_d^2 & 0 & 0 & K_a K_d & \frac{1}{2}(-K_a^2+K_d^2) & 0 & 0 & 0 & 0 & -\frac{1}{2}K_a^2 & 0 & 0 & 0 & 0 \\
0 & 0 & -K_a K_d & 0 & 0 & 0 & 0 & 0 & K_a^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2}K_a^2 & 0 & 0 & -\frac{1}{2}K_a^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2\,a^3\,\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4\,a^2\,\alpha+K_d^2 & 0 & 0 & -K_a K_d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_a^2-K_d^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_a^2-K_d^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -K_a K_d & 0 & 0 & K_a^2
\end{pmatrix}$$

$$\text{Eigenvalues[Curved]} = \left\{0,\ 0,\ 0,\ 0,\ \frac{1}{2}\left(K_a^2-K_d^2\right),\ K_a^2-K_d^2,\ K_a^2-K_d^2,\ K_a^2-K_d^2,\ K_a^2+K_d^2,\ K_a^2+K_d^2,\ f_1(K),\ f_2(K),\ f_3(K),\ f_4(K)\right\}$$

where $f_1(K)$, $f_2(K)$, $f_3(K)$,
$f_4(K)$ are functions of K such that $K^2 \neq 0$.