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## Some Examples of the Interplay Between Algebra and Topology

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# **Some Examples of the Interplay between Algebra and Topology**

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With Honors in Mathematics

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# Some Examples of the Interplay between Algebra and Topology

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Approved:

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# Introduction

Few subjects within pure math are considered more foundational than Algebra. It is the study of operations (multiplication, addition) between numbers (and often between non-numbers!) and how to solve equations which involve those operations. Historically, some of the earliest known complex math problems involved solving quadratic equations and systems of equations.

In contrast, topology is a relatively recent subject, with much of its study being formalized only within the last 150 years. Topology is concerned with large-scale properties of a space (an object) which do not change when you bend or stretch it. For example, in topology, a square and a circle are roughly the “same” because you can bend a circle into a rectangle and vice-versa. Some common properties which are studied in topology are continuity, whether or not a space is connected, and if two spaces can be bent, stretched, or smushed so that they resemble each other.

These fields may seem unrelated, but they are actually used quite often in conjunction with each other. A prime example of this is calculus. One of the forms of the fundamental theorem of calculus states

**Theorem.** *If  $f$  is a continuous function on  $\mathbb{R}$  and  $F$  is a function whose derivative is  $f$ , then*

$$\int_a^b f(x)dx = F(b) - F(a)$$

Note that you cannot have this theorem without the algebraic structure of addition on the real numbers for you could not have the expression  $F(b) - F(a)$ . However, you could also not have it without the topological structure of continuity (without it, the theorem definitely does not hold). Even just the real number system cannot exist without the union of algebraic and topological concepts. The real numbers are so powerful for many reasons, but two big ones are that it is a field and a complete metric space. A field is an algebraic structure with a well-behaved addition and subtraction and well-behaved multiplication and division which distribute over the addition and subtraction. They tend to have very nice properties and powerful theorems that apply to them. On the other hand, being a complete

metric space implies that there is some way to measure distance in your space and that your space has no holes. In particular, it means that if things are getting arbitrarily close together, then they are converging to something. This allows you to work with limits in the real numbers and guarantees the unique existence of these limits.

However, this really only exemplifies the idea that combining very deep and interesting fields of mathematics gets you more deep results. It is still difficult to see how these two fields should relate directly to each other without creating a whole new field. Yet, there are indeed many deep and interesting connections between just the two fields. You can define operations on objects in topological spaces which can characterize spaces and help to determine their properties, and similarly, you can also study the structure of a specific operation, by studying various topological spaces associated with it.

This paper explores both sides of this relationship. For using algebra to understand topology, we will examine the fundamental group, homology groups, and cohomology groups. On the side of using topology to understand algebraic structures, we will examine  $K(G, 1)$  spaces and dimensions of groups.

But before we get to that material, we will first explain the goal and modus operandi of this paper. First, the intended audience is undergraduates with little to no experience with either algebra or topology. As such, there will be few proofs and anything that does resemble a proof will be in the form of non-rigorous outline of a proof, skipping over much of the delicate argumentation needed for most of these theorems. If you wish to see some excellent rigorous versions of these arguments, it is suggested that you check out *Algebraic Topology* by Allen Hatcher (See [3]) for all of the subjects except  $K(G, 1)$  spaces and cohomological dimension. It is an excellently written book and is free on his website. For more information on  $K(G, 1)$  spaces and topological group theory, check out *Topological Methods in Group Theory* by Ross Geoghegan [2]. For cohomological dimension, you should look at Kenneth Brown's *Cohomology of Groups*, which is fairly dense but as the subject is fairly advanced, a fitting introduction to it [1].

## 1 Background in Algebra

The only algebraic structure which we will use is a group, and we will not be using many big group-theoretic results. (as a consequence, if you have taken a course in algebra, or have worked with groups in some manner, feel free to skip this section). A group is a collection (a set) of objects with a specific operation which satisfies four axioms. We will denote the operation here by the multiplication symbol  $\cdot$  (although oftentimes we will leave the symbol out, as it often is with normal multiplication). The



axioms are

1. If  $a$  and  $b$  are in your collection, then  $a \cdot b$  must be as well. (*closed under the operation*)
2. If  $a$ ,  $b$ , and  $c$  are in your set, then  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  (*associativity*)
3. There must be an object  $e$  such that for any other object  $a$ , we have  $a \cdot e = e \cdot a = a$ . This element is unique and is often denoted by the number 1. (*existence of an identity*)
4. Every object  $a$  must have an inverse  $b$  such that  $a \cdot b = b \cdot a = 1$ . (*existence of inverses*)

Some examples of groups are the integers  $\mathbb{Z}$  with the operation of addition and the rational numbers excluding 0 with multiplication.

A less obvious example is called the *free group on  $S$*  or equivalently, the *free group generated by  $S$*  where  $S$  is a collection of objects. To construct this, first we create an alphabet  $S'$  consisting of the elements of  $S$  and for each element  $a$ , an additional letter  $a^{-1}$  which will be the inverse for it. The group consists of all words (finite strings of letters) made with characters from  $S'$  and the operation is appending the words together to make a new word. However, we add the extra condition that you can cancel out any pairs of the form  $aa^{-1}$  and  $a^{-1}a$ .

Let's look at an example. Let  $S = \{a, b\}$ . Then the free group on  $S$  are all strings of letters from  $S' = \{a, b, a^{-1}, b^{-1}\}$ . Some examples of elements of the group are  $a$ ,  $aabababbbbbbbaaaa$ ,  $a^{-1}b^{-1}ab$ ,  $aaaaaaaaaaaa$ ,  $aaaaaaaaaaaa$ ,  $a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}$ , and the identity element, the empty string, “ ” (which we will denote by 1 to avoid confusion). But writing many of those strings can be incredibly cumbersome, so we often combine strings of the same letter into one letter raised to a specific power, e.g.  $aaaaaaaa = a^7$ ,  $b^{-1}b^{-1}b^{-1}b^{-1} = (b^{-1})^4$  (which we often abbreviate further to  $b^{-4}$ ). Note that with free groups, the axiom of associativity is incredibly crucial, allowing us to think of  $abba$  as  $(ab)(ba)$ ,  $((ab)b)a$ ,  $a(b(ba))$  or as any other way to split up the string.

Now, let's look at the special case when we have the free group on a single generator (let's say  $S = \{a\}$ ). Then any element can be represented by  $a^n$  where  $n$  is an integer. Even better, we can see that if we append  $a^m$  and  $a^n$  where  $m$  and  $n$  are both positive, the resulting string will be  $a^{m+n}$  (If  $m = 3$  and  $n = 4$ , then  $a^3a^4 = (aaa)(aaaa) = aaaaaaa = a^7$ ). We can even go further and see that this even holds if they are not of the same sign (if  $m = -5$  and  $n = 2$  then  $a^{-5}a^2 = a^{-1}a^{-1}a^{-1}a^{-1}a^{-1}aa = a^{-1}a^{-1}a^{-1} = a^{-3}$ ). Let us notice two things, all that really matters is the power that  $a$  is raised to, and the way you perform the operation is by adding the powers of the elements. Well, then why do we even need to write down the  $a$ ? We can see that in some way this group is the “same” as the integers with addition, except we have renamed all the integers as  $a^n$  instead of  $n$ . If two groups are the “same”

like this, that is the structure of the group is the same, but the actual objects being operated on are different, we say that the groups are *isomorphic* to each other.

One more important idea in group theory which we have to consider is the idea of a subgroup. A group  $H$  is a subgroup of a group  $G$  if every element in  $H$  is an element in  $G$  and they have the same operation. We usually denote  $H$  being a subgroup of  $G$  by writing  $H \leq G$  (or  $H < G$  if we know  $H \neq G$ ). Now, let's look at an example of subgroups. Let our big group  $G$  be the free group on two generators,  $a$  and  $b$ . Let  $H$  be the set of all of the strings of the form  $(ab)^n = ababab \dots abab$  and  $(b^{-1}a^{-1})^m = b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1} \dots b^{-1}a^{-1}b^{-1}a^{-1}$  for any  $m, n \geq 0$  (Note that this includes the empty word since  $n = 0$  is included). Now, let us construct an argument to show that  $H < G$ . It is pretty clear that every element of  $H$  is in  $G$ , so we just need to show that  $H$  is a group. We do not need to check that the operation is associative since it is associative in the bigger group (this will actually hold true for any subgroup, you never need to check if the operation is associative when determining whether a subset of a group is a subgroup). We also can see that since  $H$  includes the empty word, it has an identity element (you can actually show that the identity element of a subgroup MUST be the identity element of the bigger group, so this property is always very easy to check). Now, we need to show that every element has an inverse. If we take  $(ab)^n$  then we can see that  $(b^{-1}a^{-1})^n$  is its inverse. As an example, let's choose  $n = 2$ .

$$(abab)(b^{-1}a^{-1}b^{-1}a^{-1}) = (aba)(a^{-1}b^{-1}a^{-1}) = (ab)(b^{-1}a^{-1}) = aa^{-1} = 1$$

Now, all that remains is to show  $H$  is closed under the operation. Appending  $(ab)^n$  with  $(ab)^m$  gives  $(ab)^{m+n}$  and appending  $(b^{-1}a^{-1})^m$  with  $(ab)^n$  (regardless of the order) gives either  $(b^{-1}a^{-1})^{m-n}$  or  $(ab)^{n-m}$  depending on whether  $m$  or  $n$  is larger. So, we have checked all of the axioms, and so  $H$  is a group and is therefore a subgroup of  $G$ .

Now, let's look at another quick example. Let  $H$  be all the words of the form  $a(ba)^n = abab \dots ba$  or  $(a^{-1}b^{-1})^m a^{-1} = a^{-1}b^{-1}a^{-1}b^{-1} \dots b^{-1}a^{-1}$  for  $m, n \geq 0$ . Right off the bat, we can see that  $H$  is not a subgroup because it does not include 1. What if we redefine  $H$  to include 1? Then  $H$  is associative, has an identity element, and is closed under taking inverses. However, it is not closed under the operation since appending  $a$  to  $a$  gives us  $a^2$  which is not in  $H$ . In general when checking that a subset of a group is a subgroup, the most difficult part is to show that the subset is closed under taking inverses and closed under the operation.

## 2 Background in Topology

One of the main concepts in topology is the idea of continuity. While it has many different formulations, one of the more broadly-applicable intuitive definitions is that a function is continuous if an arbitrarily small change in output can be produced by a sufficiently small change in input. This is all we are going to say about continuity, but note that functions between topological spaces will be assumed to be continuous unless otherwise stated.

A collection of topological objects that we are going to use extensively are the  $n$ -dimensional spheres. An  $n$ -dimensional sphere is the set of all points which are distance 1 from the origin in  $\mathbb{R}^{n+1}$  and is denoted by  $S^n$ . So, using this definition,  $S^0$  is just two discrete points ( $-1$  and  $1$ ),  $S^1$  is the circle of radius 1 about the origin, and  $S^2$  is the hollow orb of radius 1 about the origin. Any sphere beyond  $S^2$  cannot be embedded in  $\mathbb{R}^3$  and so is not possible to visualize (although there are tricks to get a feel for what some of them are like). Note that because you are allowed to bend and stretch things in topology we can actually see that spheres are a little more general. For example, every polygon which is not self-intersecting is equivalent to  $S^1$  (in this case the type of equivalence we are using is called *homeomorphism*, as there are many different notions of equivalence in topology) and every regular polyhedron is equivalent to  $S^2$  because you can bend it and smooth out the edges and vertices.

An object which is very related to a sphere is a ball. An  $n$ -ball is the set of all points which are *at most* distance 1 away from the origin in  $\mathbb{R}^n$ . From this definition, the 0-ball is a single point ( $\mathbb{R}^0$  is a single point), a 1-ball is an interval, a 2-ball is a flat disk, and a 3-ball is a filled-in orb. With a little thought we can actually come up with an alternate definition for an  $n$ -sphere using balls. Each  $n$ -sphere is the boundary of an  $(n + 1)$ -ball. This idea will be incredibly important later in the sections on homology and cohomology, but for now we will just mention that most of the topological spaces in this paper will be created (or can be created) by taking a bunch of points (0-balls) and then gluing 1-balls to those points, and gluing 2-balls to those 1-balls, and so on (and we actually do not need to stop, we can continue gluing forever). A space which is created this way is called a CW-complex. At each step in the process of creating a CW-complex we create what is called an  $n$ -skeleton. The set of points we start out with is called the 0-skeleton. After attaching the 1-balls (intervals), we have the 1-skeleton, and after attaching the 2-balls, we have the 2-skeleton, and so on. So, we can see that if  $n \leq m$  then the  $n$ -skeleton of a CW-complex will be contained in the  $m$ -skeleton of the space.

And armed with these topics, we are ready to venture boldly into the world of algebraic topology!

# Using Algebra to Study Topology

There are many ways that you can conceive of using algebra to study topology. However, we are going to focus on one approach, using algebraic structures on objects in a space to describe certain structural properties of the space and to help distinguish it from other spaces. We'll start by examining the fundamental group, which is a way of making a group out of paths in your space which all start and end at the same point. We'll then move onto homology, which can be thought of as a way of counting holes in your space (even though the technical definition is a little more arcane). Then finally we will consider cohomology, which is similar to homology except you add the prefix co- to all of the concepts and reverse the directions of any arrows you see.

## 1 The Fundamental Group: The Path to Knowledge

Long after Theseus has slain the minotaur, Daedalus returns to the labyrinth...

He is intrigued how everybody gets so lost in a simple labyrinth that he has made, so he decides to map out all of the possible paths you can take in the labyrinth. He is wary though, so at the

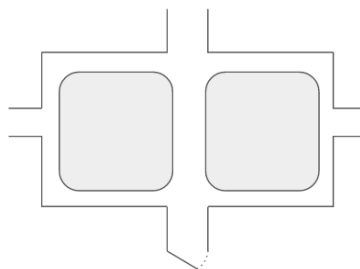


Figure 1.1: The entrance and a small portion of the labyrinth.

entrance, he makes sure to tie the string to a nearby hook on the wall. He then decides to go exploring. Eventually, he comes back to the entrance and decides for himself “Okay, my path is done!” So he cuts off the string and ties it off and labels the path as “path number 1.” He then ties a new string to the hook, before shortly realizing that he’s dropped his keys somewhere. He rolls his eyes and searches

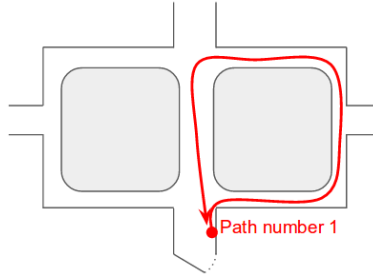


Figure 1.2: The first path that Daedalus makes.

back along path 1, before quickly finding them. So that this doesn't happen again, he decides to leave his keys on a hook on the door, but he then comes back and notices that technically he has made a path! He thinks to himself "Surely this does not count as a path!" and gets a wide-eyed expression

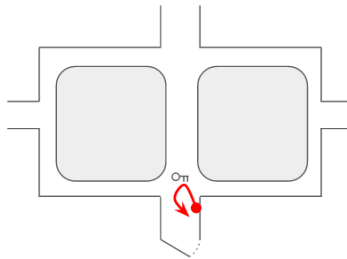


Figure 1.3: Clumsy Daedalus accidentally makes a path while retrieving his keys.

before realizing that with his current notion of path (one that requires every formation of string to be a different path), he would quickly run out of string, just by making minuscule paths in the doorway. He decides to save himself some time by declaring that two paths are equivalent if after you've tied them both off, you can move their strings so that they lie right next to each other.

Shortly after coming up with this idea, he starts another path and comes back to the start and ties it off and labels it "path number 2." He then realizes that he could take the end of path 1 and the start of

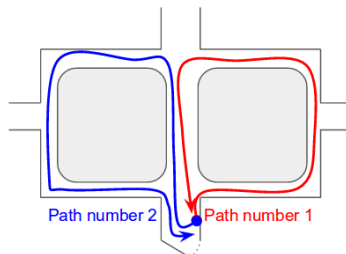


Figure 1.4: The first two paths what Daedalus makes.

of path 2 and tie those together and he would get a completely different path! So, he unties the end of the first path and the start of the second path from the hook, ties the two together and labels it "path 1

+ path 2.” Daedalus has made an operation on paths in his space, where the operation takes two paths

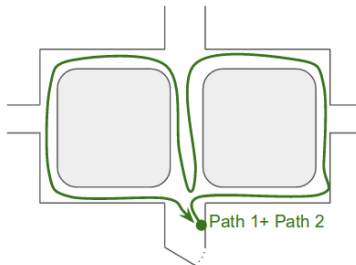


Figure 1.5: The result of performing his operation on path number 1 and path number 2.

and concatenates them (but after concatenating them, you are allowed to freely move the midpoint). This operation actually satisfies all of the axioms of a group! What group might you ask?

## 1.1 Formal Definition of the Fundamental Group

The definition of the fundamental group is certainly more formal and rigorous than in the above story, but Daedalus was definitely onto something. First, let us define a *path* in  $X$  as a continuous function from an interval to  $X$ . This essentially allows a path in your space to be any interval embedded in  $X$ . In order to define the fundamental group of a space  $X$ , you must first choose a point  $x$  in  $X$  as a basepoint. If there is a path between any two points in  $X$  (the space is path-connected), as it will be throughout this paper, then your choice of basepoint does not affect the resulting structure of the group (the resulting groups will be isomorphic). Now, we start by taking all the paths in your space which start and end at  $x$  and define two paths to be equivalent if you can *continuously deform* (move) one path to the other while keeping the ends of the loop fixed at  $x$ . If two paths are equivalent like this, we say that they are *homotopic relative to  $x$*  (or if it's clear from the context, we simply say homotopic). The operation on these paths is concatenation, so for two paths  $a$  and  $b$  based at  $x$ ,  $ab$  is the path where you go along  $a$  first and then  $b$ . Note that  $ab$  only needs to keep *its* endpoints fixed. The end of  $a$  and the beginning of  $b$  is now allowed to wander freely to determine whether two curves are the same. One way you can accomplish this concatenation is by appending the intervals each path is mapped to. That is, if the domain of path  $a$  and  $b$  are  $[-\pi, 4]$  and  $[e, 3]$  then you can perform a change of variables on the second path to get a new interval on which  $ab$  is defined, namely  $[-\pi, 4 + (3 - e)] = [-\pi, 7 - e]$  (note that the length of this interval is the sum of the lengths of the original intervals). Using changes of variables, we could even go further and require that the domains of all paths be  $[0, 1]$ . When you are appending two paths with this restriction, make the first go twice as fast on  $[0, \frac{1}{2}]$  and the second go twice as fast on  $[\frac{1}{2}, 1]$ .

But are we sure that the fundamental group is actually a group? Let's check the axioms. We can see that it's closed under the operation (If you concatenate two paths which start and end at  $x$ , you will get a path with start and point at  $x$ ). It is not too difficult to see that if you concatenate three paths together, it does not matter whether you concatenate the first two before you concatenate the last one or concatenate the last two before concatenating it onto the first one, so the operation is associative. We can see that the constant path (the one which stays at  $x$ ) is the identity because concatenating it to another path does not actually change the path at all. (Note that this means that any path which is homotopic to the constant path is also the identity). And we can also see that if we concatenate the reverse of any path to itself, we can move the middle-point back along the first path to  $x$ . Therefore

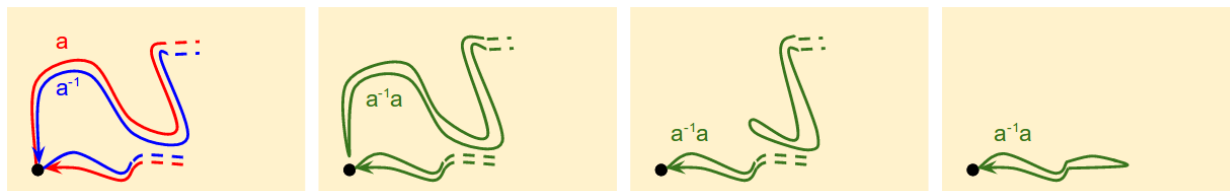


Figure 1.6: A path and its inverse being deformed to the constant path.

the fundamental group is indeed a group.

The fundamental group of the space  $X$  with basepoint  $x$  is denoted by  $\pi_1(X, x)$ , but often you are unconcerned with which point you choose as basepoint, in which case you can suppress the  $x$  in the notation and write  $\pi_1(X)$ .

## 1.2 Some Examples of the Fundamental Group

Now, let's work out some examples of the fundamental group

**Example:**  $\pi_1(S^2)$

It's certainly plausible that any loop on the sphere can be deformed to be the constant map (just try putting a hair-tie on a grapefruit and you will see that it is quite plausible). Thankfully, our intuition is right and it actually does turn out that every path is homotopic to the constant map, but a rigorous proof of this is much more difficult and involves a lot of careful argumentation (mainly because of the fact that your curve could potentially go through every point in the sphere). But because this indeed hold true, the group consists only of the identity element, also called the trivial group.

**Example:**  $\pi_1(S^1)$ 

If a path on the circle which is based at a point backtracks on itself, then we can continuously deform it to smooth the path out (harkening back to the Daedalus example, start pulling on one of the strings at the door knob until the string is taut). So each path is determined by how many times it wraps around the circle and in which direction. Let's denote the path that wraps around the circle clockwise  $n$  times as  $a^n$  and one that wraps around counterclockwise as  $a^{-n}$ . Now, if we concatenate two loops of the same direction, they add, i.e.  $a^n a^m = a^{m+n}$  and  $a^{-n} a^{-m} = a^{-(n+m)}$ . And, if we concatenate two loops in opposite directions, they will cancel out (think again of pulling on that string), so we get  $a^m a^{-n} = a^{m-n}$ . Therefore, we can see that this is isomorphic to the integers.

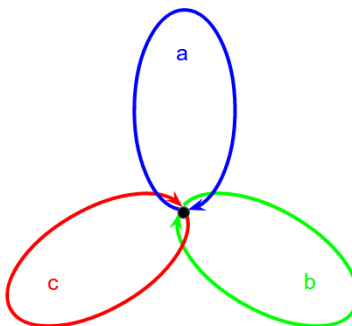
**Example: A Bouquet of circles**

Figure 1.7: A bouquet of 3 circles

A bouquet of  $n$  circles is  $n$  copies of  $S^1$  all attached together at one point. If we choose our basepoint to be the point where all the circles are attached together, we can choose any of the circles and a direction to go on before coming back to the original point. So, every path is of the form  $b_1^{n_1} b_2^{n_2} \dots b_m^{n_m}$  where each of the  $b_i$  represents one of the circles (you can think of this as traversing loop  $b_1$   $n_1$  times then  $b_2$   $n_2$  times and then doing this as many times as you want since  $m$  can be as large as you want). Here we are also thinking of each  $n_i$  as being any integer, since you can think of as a negative  $n_i$  as going in the opposite direction as  $-n_i$ . So, with a little bit of thought, it is clear that this is the free group on  $n$  generators where  $n$  is how many circles there are in the bouquet.

We will cover some more examples, but first we will state a theorem which will help us to determine fundamental groups.



### 1.3 The Pancake Theorem

I have not found an actual name for this next theorem (from [3]), so I have christened it the pancake theorem for easy reference later on. The theorem describes what happens to the fundamental group of a space when you glue a disk (2-ball) onto it, which is akin to trying to glue or sew a pancake onto an object.

**Theorem** (The Pancake Theorem). *If  $X$  is a space and  $G = \pi_1(X)$  then if you attach a disk onto  $X$  along a path  $f$  in  $\pi_1(X)$  to obtain  $X'$  then  $\pi_1(X')$  is  $G$  with the additional relation that  $f = 1$ .*

Now, in order to understand this theorem, we need to understand what a relation is. In a group, a relation is a way that two elements of the group are related. For example, if we let  $G$  be the free group on two generators  $a$  and  $b$ , with the additional relation that  $ab = ba$ , then whenever we see an  $ab$  in a formula, we can replace it with  $ba$ . This actually means that we can switch around the orders of  $a$  and  $b$  as we like, so we can actually simplify any of the words down to  $a^n b^m$  for some  $m$  and  $n$  in the integers. Let's look at an example of how to do this. For example, let's show  $abaaab = a^4 b^2$ , by using the relation  $ab = ba$ .

$$abaaab = a(ba)aab = a(ab)aab = aa(ba)ab = aa(ab)ab = aaa(ba)b = aaa(ab)b = a^4 b^2$$

This way of thinking of a group as some other group with extra relations is an extremely powerful mode of thought. So much so, in fact that you can show that every group is isomorphic to a free group with additional relations. A specific representation of a group as a free group with some relations is called a *presentation* of the group. The way that you normally specify a group presentation is  $\langle \text{generators} | \text{relations} \rangle$ . So, one way that you can specify the  $G$  that we made is by saying  $G = \langle a, b | ab = ba \rangle$ . We can now restate the Pancake Theorem as

**Theorem** (The Pancake Theorem 2.0). *If  $X$  is a space and  $G = \pi_1(X) = \langle \text{generators} | \text{relations} \rangle$  then if you attach a disk onto  $X$  along  $f$  in  $G$  to obtain  $X'$  then  $\pi_1(X') = \langle \text{generators} | \text{relations}, f = 1 \rangle$ .*

Now, we will provide a partial proof of a part of this.

*Sketch of Proof in One Direction.* We can see that after the disk is attached, the boundary of that disk can now move through the disk down to a point, i.e. the constant map.  $\square$

This is a pretty good explanation for why you get the added relation, but it is far from complete. How do you know that you do not get any additional relations? That's where you get to the difficult part. The part is difficult enough that if you are very curious we will direct you to [3].

## 1.4 Some Applications of the Pancake Theorem

Now, with this new tool in hand, we can tackle bigger and better fundamental group calculations.

### Example: The Torus

The torus is a topological space which is roughly like an inner-tube or a hollowed out donut. There are several ways to construct a torus, but for the purpose of calculating the fundamental group using the pancake theorem, we will construct it by attaching a disk to a bouquet of two circles. By an earlier example, we know that so far, our space has a fundamental group isomorphic to the free group on two generators,  $a$  and  $b$ . Now, in the diagram, we can see that we can construct the torus by gluing a disk

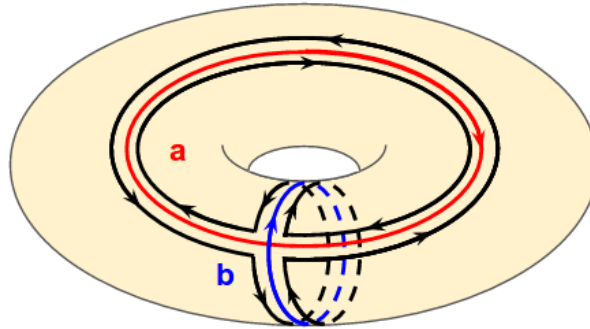


Figure 1.8: A construction of the disk by gluing along  $aba^{-1}b^{-1}$  (the boundary of the disk is the black curve)

onto  $aba^{-1}b^{-1}$ . So, the resulting presentation for our group is  $\langle a, b | aba^{-1}b^{-1} = 1 \rangle$ . But, we can simplify the relation  $aba^{-1}b^{-1} = 1$  by multiplying the equation on the right by  $ba$  to get  $ab = ba$ . We learned earlier that each element in this group depends only on the power of  $a$  and  $b$ . And you can further show that the way you multiply two elements is by adding the corresponding powers of  $a$  and  $b$ . This group is isomorphic to a group called  $\mathbb{Z}^2$  which is represented by pairs of integers  $(m, n)$  and the operation is item-wise addition i.e.  $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$ .

### Example: The Real Projective Plane $\mathbb{R}P^2$

Another way to construct the torus is to take a square and glue together opposite edges. To see why this is true, we do it one step at a time: we first get a cylinder, and then what used to be sides of the square are now the holes in the cylinder, and so you glue those holes together and get a torus.

The real projective plane is a topological space where each point is surrounded by a normal disk, but the overall structure is incredibly finicky (you can't even embed the real projective plane in  $\mathbb{R}^3$  without it intersecting itself). One way to construct the real projective plane is by taking a square and

gluing opposite edges together in the opposite direction. What does this mean? If you do it step-wise just like the torus, you will first get a Möbius strip. However, you now have to try and think of a way to glue together the boundary of the Möbius strip, which is very hard to visualize. So, let's try a different method. You can also create the real projective plane by taking a sphere and gluing antipodal (opposite) points together.

The final way we will discuss is by gluing a disk onto a circle. However, we have to do it in a funky way. Suppose  $a$  is the generator for the fundamental group of the circle, then we glue the disk along  $a^2$ . One way to try and visualize this is to think about double-wrapping a rubber band, but instead of using a hollow rubber band, you use a filled-in sheet of rubber and then attach each of the circles of the double-wrapped rubber sheet onto the same circle. This approach has the advantage that it makes calculating the fundamental group very simple. We can see that  $\langle a | a^2 = 1 \rangle$  is a presentation for the group. What does this group look like? Well, we can see that whenever we have  $a^n$ , we can keep on canceling out pairs of  $a$ 's until we get  $a^1$  or  $a^0 = 1$ . This gives us the strange realization that there are only two paths which are not equivalent, the constant map and some other curve. This other curve also has the property that when you append it to itself, it is equivalent to the constant map. Using the first definition of the real projective plane, we can actually see what this curve is and why it has these properties (hint: the curve bisects square before you glue it).

## 1.5 Final Thoughts About the Fundamental Group

The fundamental group is a very powerful tool for telling when two topological spaces are not equivalent (In this case, equivalent means homotopic, i.e. you can stretch, bend, squish, expand, or move the space through itself to determine if the spaces are equivalent). However, there are many spaces which do have the same fundamental group but which are different. It also has further drawbacks, one of which is that many of the ways in which you find the fundamental group give you presentations of your group which depend on how you constructed your space, and as we just saw with the real projective plane, there can be very distinct ways of constructing the same topological space. A single group can have very different presentations and it is actually algorithmically impossible to tell whether two presentations represent the same group [5].

However, for all its drawbacks, there are some cool things you can do with it. First of all, for a given group  $G$ , you can create a topological space with fundamental group  $G$ ! One way you can do this is to start by taking a presentation for  $G$ , and for every generator in the presentation, choose a circle in a bouquet of circles. Then for each relation, just attach a disk along the given path. This will not make

a pretty space for you in many cases, but it certainly does the job. It also shows you that if you know everything about topology, you know everything about group theory, just by using the fundamental group!

## 2 Homology Groups: All About Vicious Cycles and Knowing your Boundaries

After Daedalus finally figures out the fundamental group of the labyrinth (for those interested, it turned out to be  $\langle a, b, c, \dots, w | bg^7 = k^2, t^2 = 1 \rangle$  which is probably why so many people kept on getting lost) he decides that this whole idea of paths being made up of other paths in a specific order is too complicated. “It took me a really long time to figure out the fundamental group for this space, I need something which is much easier to calculate.” It looks like Daedalus is starting to see some of the drawbacks of the fundamental group.

Then Daedalus, being rather bored of King Minos’s dusty old basement, decides to set his sights on loftier ambitions. Specifically, he decides to build a spaceship so that he can test out his new-found fundamental group on the heavens. Only when he gets there, he finds out that it doesn’t tell him much. With all of the wonder of the cosmos at his hands, most paths that he can make can be deformed around all of the planets, stars, and space dust into trivial loops. This disappoints him and so he needs to come up with something a little more powerful. Yet another drawback.

Maybe there is a better way...

### 2.1 Motivation for the Homology Group

Daedalus was right to be concerned by some of these problems. The fundamental group only really helps you figure out the 2-dimensional structure of your space. An intuitive explanation of this is that if we have a CW-complex, any path which goes through a high-dimensional ball can be moved to the boundary of that ball, which is contained as a part of a slightly lower dimensional ball, and then we can repeat this process until we get down to the boundaries of 3-dimensional balls to spheres. This means that we can add some very sophisticated and complicated structure as we are building up a CW-complex, but after we reach the 2-skeleton, the fundamental group of the space won’t change.

Now, there is a generalization of the fundamental group to higher dimensions called homotopy groups, where instead of concatenating loops together you concatenate  $n$ -dimensional spheres together. However, these are so difficult to calculate that most of the homotopy groups for most of the spheres

are unknown. [4]

These two drawbacks are remedied by the homology groups. The homology groups are so easy to work with that they can be calculated by computer, but are still versatile enough that they can give you information about the  $n$ -dimensional structure of an object for any  $n$  in the natural numbers.

However, there is one major drawback; the formal definition for the homology group is significantly more complicated and requires a significantly larger background in abstract algebra than the fundamental group. Though, by the end of the chapter we will arrive at a definition which preserves the spirit of the homology groups and which does not require a semester-long course in preparation.

## 2.2 Intuitive Notions of Some Homology Groups

Before that, we will start with some intuition behind the 1st and 2nd homology groups.

### The First Homology Group

The homology group has a strictly looser definition of equivalence than homotopy, which means that we have a good place to start in the 1-dimensional case. In the first homology group, like in the fundamental group, we start out with oriented loops in your space. However, there are some relaxations to the definition of equivalence used for fundamental group. There is no basepoint or fixed points. You can move your entire loop around your space as long as you do it continuously and the type of the loop will not change (the homology class of the loop). You can also have multiple loops in your space at once which are a single element of your group. To distinguish a single group element with two (or more) loops from multiple group elements consisting of one loop each, we may call the single element a “multiloop.” We then add one final major relaxation: You are allowed to cancel any portions of a path which have opposite direction in order to combine or break apart loops. One consequence of this last relaxation is that for any loops which bound disks, you can squeeze the loop down to a line and cancel out the loop with itself! If you think about it like this, your loops sort of become like explosive



Figure 2.9: A curve which bounds a disk can cancel with itself.

amoebas, splitting apart by mitosis, sometimes popping if they surround nothing, and recombining,

either by eating each other or by melding at will.

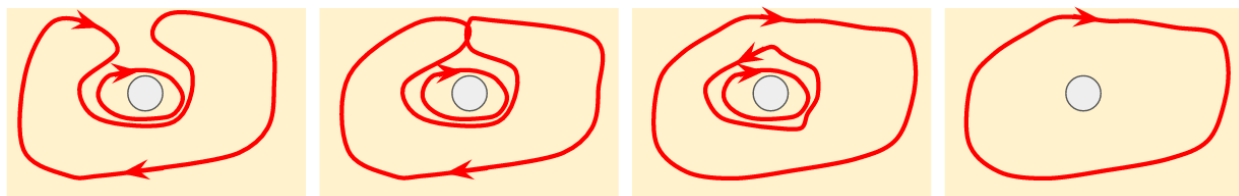


Figure 2.10: A curve in the homology group using phagocytosis to consume another curve.

Now, we were also promised that this would be a group, so what is the operation? The operation here is taking two loops or multiloops and then taking the union of those two sets of loops. The operation is associative because if you have three collections of loops, it does not matter which two you combine first, you will still end up with the same collection after all the combining is over. We can see that in this case, the set with no loops acts as the identity element. There are also inverses, because for a given bunch of loops, just take each loop with reversed orientation and then cancel out all of the paths. One really nice result of this is that we end up with a group where the order of the elements in an operation doesn't matter, which makes the group much easier to work with.

Now, let's see an example of this. Consider a disk with two holes punched out. We can see that no matter how complicated we make a loop in this space, we can pinch it off around each hole until we are just left with a variety of loops with various orientations around them. We then cancel out any loops around the same hole with opposite orientations, leaving only a bunch of loops oriented in a single direction around each of the holes (although loops around different holes might have different orientations). We can represent these loops by a pair of numbers  $(m, n)$  where  $m$  represents the number of



Figure 2.11: Simplifying a complex curve to one which is just a collection of loops around the holes.

loops around the first hole (with the sign of  $m$  representing the orientation of the loops, positive for counterclockwise and negative for clockwise) and  $n$  similarly represents the number of loops around the second. The operation in this case amounts to adding the pairs term by term. If you add 3 counterclockwise loops around the first hole to 2 clockwise loops around the first hole, we can cancel the two clockwise loops with two of the counterclockwise loops and you are left with 1 counterclockwise loop (or in our pair notation  $(3, 0) + (-2, 0) = (1, 0)$ ).

Now, we can generalize this argument to a disk with  $n$  holes. We would get  $\mathbb{Z}^n$  which gives us another interesting way to view the 1st homology group as a method for determining how many holes your space has (a view which can actually be expanded to higher dimensions!).

If we consider another example, a bouquet of  $n$  circles, we can actually use the same logic as above to compute that the homology group is again  $\mathbb{Z}^n$ . Comparing this to the fundamental group of the space, the free group on  $n$  generators, we can represent  $\mathbb{Z}^n$  as a free group on  $n$  generators with the added relation  $ab = ba$  for any two elements  $a$  and  $b$ . Because of this, we can change around the order of the elements in a word as much as we like in order to group all of the same elements into a specified order. For example, if our original space was a bouquet of 26 circles, we could represent each loop by a letter in the English. Then since the order of our letters doesn't matter anymore, we could alphabetize any word we come across without changing the group element. So for example, *cataract* =  $a^3c^2rt$ . When you take a group and add all the relations necessary to ignore the order of the elements like this, the resulting group is called the "abelianization" of the original group (a group where the order does not matter is called an abelian group). It turns out that for any space, the 1st homology group is the abelianization of the fundamental group. It also explains the fact that  $\mathbb{Z}^n$  is also called the free abelian group on  $n$  generators, since it is just the abelianization of the free group on  $n$  generators.

To see why it is plausible that the first homology group is the abelianization of the free group, let's start by examining the fundamental group  $\pi_1(X, x)$  for an arbitrary space  $X$ . If we look at two of elements  $p$  and  $q$  of the group, we can see that if we look at  $pq \in \pi_1(X)$  as an element of the homology group (it is an oriented loop after all), we can break it up into the multi-loop containing  $p$  and  $q$ . Similarly, we can see that the same holds true for  $qp \in \pi_1(X)$ , in the homology group, it can be looked at as the multiloop containing  $p$  and  $q$ . Therefore  $qp$  is equivalent to  $pq$  in the homology group (in this case we say  $pq$  is homologous to  $qp$ ). Now, what this explanation doesn't consider is the fact that it is quite plausible that when you convert from the fundamental group to the homology group, you might have to add some relation *not* of the form  $ab = ba$ . Indeed, in higher dimensions, all of the homotopy groups are already abelian, and so if you add those relations, you would just get back your original group. However, there are many cases where the homotopy group and homology group do not coincide, so it is not generally the case that the  $n$ -th homology group is the abelianization of the  $n$ -th homotopy group. In fact there are some cases when the  $n$ -th homotopy group is trivial but the  $n$ -th homology group is not!

But now it is time to leave the cushy 1-dimensional world and study the 2nd homology group!

## The Second Homology Group

In many ways, the second homology group mimics the first homology group except that you use orientable surfaces instead of circles. One concept which does not immediately extend to higher dimensions is the idea of an oriented surface. (Note that in the following section, we will not be talking about non-orientable surfaces, so no Klein bottles or  $\mathbb{R}P^2$ s!). One simple way you can choose an orientation for a surface is by embedding a working clock in it at every point. Now, this clock may not be standard, it can either move clockwise or counterclockwise and the direction the clock is spinning is the orientation of the surface. We also make the added condition that clocks which are close to each other must be rotating in the same direction in order to make the orientation consistent. Note that the way you are looking at the clocks might depend on where you are standing. For example, if you are standing inside a sphere, the clocks might look like they are moving clockwise, but as soon as you leave the sphere, they will look they are going counterclockwise. So, you can initially think of an element of the second homology group as a bunch of blobs in your space wearing (possibly backwards) watches. However, when two blobs touch, if their watches are moving in opposite directions (if their hands get stuck on each other), that portion of the surface cancels out and they connect! In fact a surface can even do this with itself. Because the orientation is consistent across the entire surface, if you take two points on the opposite side of a sphere, the clocks for those points will be moving in opposite directions relative to each other. So, just like in the 1-dimensional case, you can cancel out any blobs which bound solid 3-dimensional structures by moving the surface through the structure until all of the clocks meet and cancel each other out. So, now the picture in your head might be of a bunch of very time-concerned blobs which meld and split apart and can pop if they bound solid 3-dimensional structures.

Now, armed with this intuition, let's examine the 2nd homology group of  $\mathbb{R}^3$  with two 3-balls taken out (Note that this is intentionally similar to the example for the 1st homology group). We can see that any blob in the space can be pinched off around one of the holes until you are just left with a collection of directed spheres around each hole. After this, arguing that the second homology group of this space is  $\mathbb{Z}^2$  is identical to the corresponding argument for the first homology group.

Now, there are further ways that things can be homologous to each other once you get past the first dimension, but they are significantly more complicated. You also get the additional complication that some of the elements of the homology group are not spheres and so are not present in the homotopy group (this is how you can get a situation where a space has a trivial  $n$ -th homotopy group but a non-trivial  $n$ -th homology group).

So, with those last two statements, it might seem like homology and homotopy are very unrelated



past dimension 1. However, we can see that there is some connection because every way of embedding an  $n$ -sphere in your space will result in an element of your  $n$ -th homology group. Because of these disparities, it might be a good to just continue onto a more formal definition of the homology groups.

## 2.3 A Slightly Formal Definition of the $n$ -th Homology Group

Before, we get into the actual definition of the group, we need some definitions. Note that for a couple of these definitions, the terminology is based on the 1st homology group. Also note that in the following section, most of the terms have the word “singular” in them. This is because this section defines a type of homology called “singular homology.” However, because we have no other type of homology to confuse it with (yet), we will often not include the word “singular.”

### Singular $n$ -Simplexes

A *singular  $n$ -simplex* is a way of embedding an  $n$ -ball into your space with an added orientation. So, a singular 0-simplex is a point in your space, a singular 1-simplex is a path in your space, a singular 2-simplex is a disk embedded in your space and so on. The way we add an orientation to these is complicated, so for now just be content with the fact that orientable  $n$ -dimensional objects can have exactly 2 orientations (the orientation of a 0-simplex is thought of as positive or negative).

### Singular $n$ -Chains

A singular  $n$ -chain is a collection of  $n$ -simplices, possibly with repetition (you can include the same singular  $n$ -simplex multiple times). We again say that you can cancel out any singular  $n$ -simplices which are in the same place but which have opposite orientation (but note that we cannot deform our simplices so that they lie in the same place in order to cancel them). In a way, an  $n$ -chain is a set of simplices with some sort of “signed repetition.” For a particular space  $X$ , the set of all  $n$ -chains forms the  $n$ -th singular chain group, denoted by  $C_n(X)$ . This is a group with the operation of combining the collections of simplices (by the same argument as in the first homology group example).

### Boundaries

The *boundary of a singular  $n$ -chain* is a collection of singular  $(n - 1)$ -simplices which together form the boundaries of all of the  $n$ -simplices in your chain (again, with possible repetition). Note that when we take the boundary of a disk, we can have it determine the orientation of its boundary using the direction the watch is traveling in. Similarly, the orientation of a path determines the orientation of

its endpoints, the starting point is negative and the ending point is positive. This boundary is often thought of as a function  $\partial$  from  $C_n(X)$  to  $C_{n-1}(X)$ . An element of a chain group is called a *boundary* if it is the boundary of an  $n$ -chain of 1 dimension higher. We sometimes use the notation *n-boundary* to specify that the boundary is an element of  $C_n(X)$  as opposed to  $C_p(X)$ . So  $x$  in  $C_n(X)$  is a boundary if there is some  $y$  in  $C_{n+1}(X)$  such that  $\partial y = x$ . Note that a 1-boundary is just a collection of loops which can all be shrunk to points (they arose as the boundary of disks in the space, so you could move them through the interiors of those disks down to points) and a 0-boundary is just a pair of points with a path which connects them (that path is a 1-ball).

## Cycles

A *cycle* is an  $n$ -chain which has no boundary, i.e. a chain  $x$  in  $C_n(X)$  is a cycle if  $\partial x = 0$  (where 0 is the empty chain). This happens if all of the boundaries cancel out. In the case of the first homology group, the 1-cycles are exactly collections of loops in your space (this is assumedly where the terminology “cycle” comes from). In higher dimensions it can be a bit trickier, since you have a larger class of things which can have empty boundary. For example, if you have an 2-chain which is made up of a bunch of disks which together form a torus, the boundary of the chain is empty because all of the boundaries cancel each other out. Note that boundaries are a special type of cycle, e.g. if you take a 1-chain, its boundary will form a collection of paths whose endpoints cancel with the starting point of a path in front of it. Similarly, a collection of disks will have a collection of paths as its boundary, all of which have endpoints which cancel with another paths endpoint. Or, thinking in terms of the boundary map, we get that  $\partial(\partial(x)) = 0$  because the boundary of a boundary has no boundary (i.e. is a cycle). From this, we get that the set of all  $n$ -boundaries is a subgroup of the set of all  $n$ -cycles is a subgroup of the  $n$ -th chain group.

## The n-th Homology Group

Now, we can get to the actual definition of the  $n$ -th homology group. We first start out by taking all of the  $n$ -cycles. Then we declare that any  $n$ -boundaries are 0 in a process called *modding out* or *quotienting* by the  $n$ -boundaries. (Whenever you quotient by a set of elements in a group, for every element  $x$  in the set, you add the relation  $x = 0$ ). The resulting group is called *the n-th homology group of X* and is denoted by  $H_n(X)$ . Now, let’s see how this relates back to our intuitive definitions of the first and second homology groups.

Our intuition for  $H_1(X)$  started us off with collections of loops in the space. The formal definition starts with all 1-cycles which are exactly collections of loops. So far so good. Then, we allowed ourselves

to continuously deform loops as much as we want. If we think about a deformation as a sequence of little bumps, the area that the loop travels over to make the bump is a small disk, so the loop which surrounds it is a boundary. We can then orient the loop which surrounds the bump so that adding it to the original loop creates the new loop. We can intuitively see that *every* continuous deformation can be thought of as a sequence of small bumps, we can see that if you can move one disk to another, then they are equivalent using the formal definition of  $H_1(X)$ . In the other direction, we can see that if you

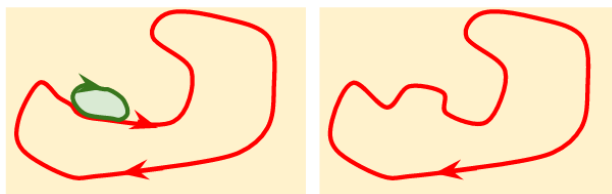


Figure 2.12: One of the bumps in an approximation of a continuous deformation by a sequence of bumps.

add a boundary to a loop, you could have just moved the loop across the interior of that boundary, and so if two loops are homologous, then they can be deformed to each other. And we can further see that we can cancel oppositely-oriented boundaries as before. As for  $H_2(X)$  we can use much of the same type of argumentation as for  $H_1(X)$  (you can move two spheres together if and only if you can perform a series of bumps across small 3-balls to get the other).

We can also see here that our view of the homology group as a way of counting holes is supported by the formal definition. Because a hole is really a cycle (an object without boundary) which does not bound something filled in. So,  $H_1(X)$  counts the number of 2-dimensional holes (it is all of the loops which do not surround disks),  $H_2(X)$  counts the number of 3-dimensional holes (it is all the balloons which do not surround solid blobs) and so on.

## 2.4 Simplicial Homology

Now, singular homology, while quite complicated, is general enough for any topological space. However it lacks a way to calculate it easily. This is where simplicial homology comes along. In singular homology, you start off with uncountably-infinitely generated chain groups and then you declare that all the boundaries (which are also uncountably generated) are 0. This sounds like a mess. You start off with uncountable-dimensional groups and then quotient by an uncountable-dimensional subgroup and then you often end up with a finite-dimensional group. Luckily, there is an (often) finite-dimensional alternative called *simplicial homology*. You start off with a finitely generated chain complex and it ends up being easy enough to calculate that a computer can do it.

Now, in a similar vein to before, we will start off with some definitions.

## Simplex

An  $n$ -simplex is a generalization of a triangle to  $n$ -dimensions. A 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. To get an arbitrary  $n$ -simplex, take all points  $(t_1, t_2, t_3, \dots, t_n, t_{n+1})$  such that each  $t_i \geq 0$  and  $t_1 + t_2 + \dots + t_n + t_{n+1} = 1$ . However, our examples will only use the four simplices mentioned explicitly above, so the general formula will not be needed.

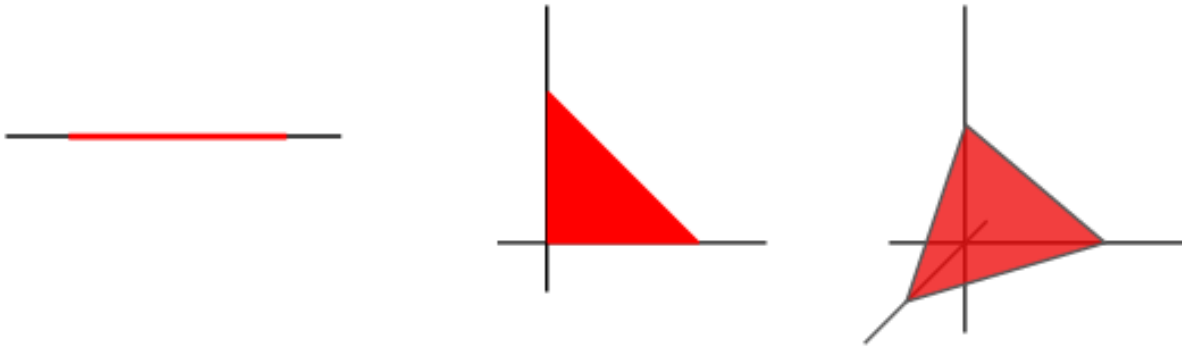


Figure 2.13: All of the easily visualizable simplices (except the 0-simplex, a single point).

Now, in order to do simplicial homology, you need to be able to split up your space into a collection of simplices. We have to do this so that adjacent simplices touch only along faces. So, for example, we cannot have an edge touching the interior of a triangle and we cannot have a point in the middle of an edge. Note however, that since we are in the bendy world of topology, we allow our triangles to be a little bent (as in figure 2.14). The process of doing this is called triangulating your space. Once you have done this, you can define a simplicial  $n$ -chain.

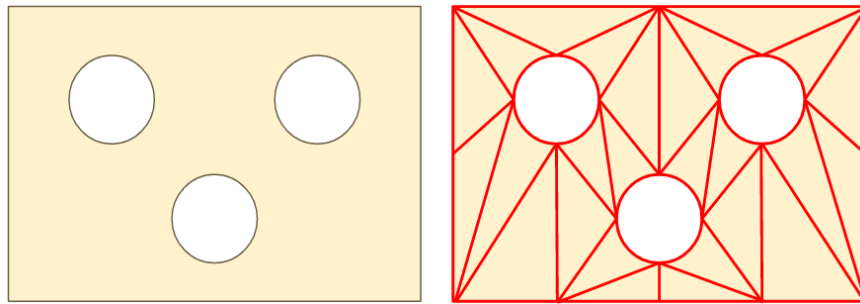


Figure 2.14: A space and a triangulation of the space.

## Simplicial $n$ -chains

A simplicial  $n$ -chain is collection of oriented simplices in your space (including repetition). The definition for an orientation on a simplex is a bit more concrete than for general  $n$ -dimensional bodies. An orientation on an  $n$ -simplex is just an ordering of its faces, and two orderings are the same if you can move them through  $\mathbb{R}^n$  so that the numbers match. For example, if you number the four faces of a tetrahedron then you can rotate it so that the 1 faces away from you. Then you can rotate

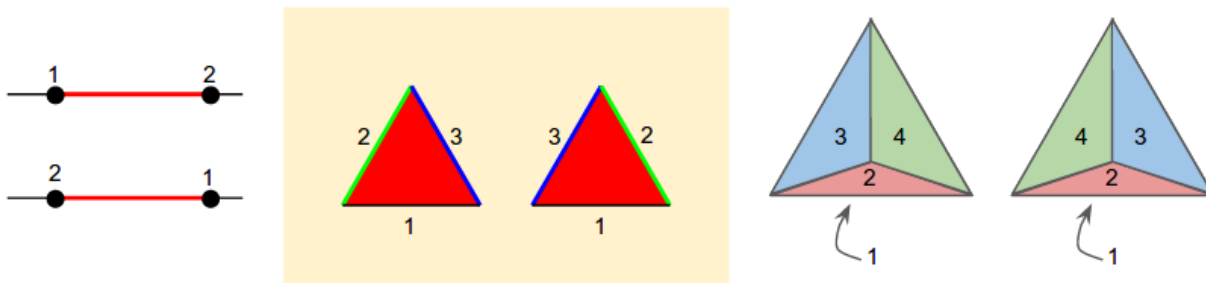


Figure 2.15: The two orientations of some common simplices (for the tetrahedra, the 1 is facing away from you).

the tetrahedron around the face with 1 on it so that the 2 lies on the bottom. So, after that either the 3 is on the left and the 4 is on the right or vice-versa. Whether or not the 3 is on the left determines orientation of the simplex. From this definition, there should be exactly two orientations for a simplex. There is one special case for oriented simplices, namely a 0-simplex should have two distinct orientations. So, we again arbitrarily define that points can either be positive or negative.

Now, in order to create a simplicial 2-chain for example, if you triangulated your space into some number of triangles, you could choose any of those triangles as many times as you would like with whatever orientation you want. Note however, that we still cancel out any copies of the same simplex which are oriented oppositely. So, if you were to assign a default orientation to each simplex, you could then just assign an integer to each simplex denoting the orientation (positive if it agrees with the default orientation, negative otherwise) and number of copies of the simplex. This way of handling orientations will be used throughout most of the examples. Your resulting chain group still has infinitely many elements (a copy of  $\mathbb{Z}$  for each simplex), but if you triangulate your space into finitely many simplices, (which is what we will be doing here) you will get a finitely generated free abelian group.

## Boundaries of Simplices

The boundary of an  $n$ -simplex  $x$  is the collection containing all of its  $n+1$  faces where a face is one of the  $(n-1)$ -dimensional simplices which bounds  $x$ . So, the boundary of a tetrahedron is the collection of triangles which bound it, the boundary of a triangle is its 3 edges, the boundary of a line segment are its two endpoints, and for reasons not fully explained here, the boundary of a point is either considered to be empty or the number 0. In general, a face is defined by fixing one of the coordinates at 0 in the parametrization above, but again no examples here will require general simplices. Note that when you take the boundary of a simplex it determines the orientation of its boundary pieces. For a triangle, this is not too difficult to see, choosing an orientation results in choosing either clockwise or counterclockwise, and then you just order the endpoints of the line segment so that you increase as you go with the direction of the rotation (see figure 2.16). For a line segment, you can just choose the

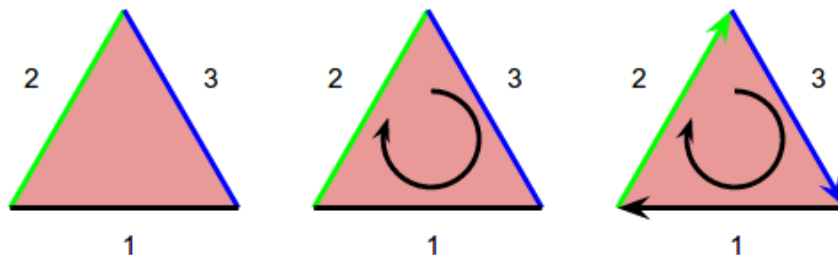


Figure 2.16: When taking the boundary, the orientation of the faces of a triangle are determined by the orientation of the triangle

initial point to be negative and the end point to be positive. The orientation for a tetrahedron is much harder, but thankfully we will not need the actual definition, we will just need a general fact about the orientation. For two neighboring simplices (simplices which share a face), the face that they share will be oriented the SAME in their respective boundaries if and only if the simplices have OPPOSITE orientations. So, if you have two simplices  $x$  and  $y$ , and they share face  $f$ , then in the boundary of  $x$ ,  $f$

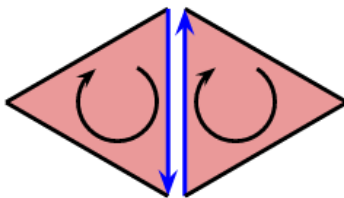


Figure 2.17: For two adjacent simplices with the same orientation, any faces that they share will be oppositely oriented in their respective boundaries.

will be oriented the same for both  $x$  and  $y$  if and only if  $x$  and  $y$  have opposite orientations. One final

note about the boundary of a  $n$ -chain is that if you have  $m$  copies of a simplex in an  $n$ -chain, then in the boundary, you will get  $m$  copies of each of the  $n + 1$  faces.

### **Simplicial $n$ -Cycles**

Now that a simplicial  $n$ -chain and its boundary have been defined, the definitions of  $n$ -cycles and  $n$ -boundaries follow naturally. A simplicial  $n$ -cycle is an  $n$ -chain whose boundary is empty. That is, when you take the boundary of a simplicial  $n$ -chain, all corresponding boundary faces cancel each other out (or equivalently, all of the simplices have the same orientation and number as their neighbors!). For example, a simplicial 1-cycle is a sequence of lines all oriented in the same direction so that all initial points cancel out with endpoints of other lines, i.e. a sequence of directed loops made out of 1-simplices in the triangulation. Similarly, a simplicial 2-cycle is a collection of triangles where every triangle has three neighbors which are all oriented the same as it, a triangulation of an oriented surface. Another way of saying that an  $n$ -chain is an  $n$ -cycle is to say that the chain is sent to 0 through the boundary map. So, by our definition of boundary of 0-simplices, 0-cycles and 0-chains are exactly the same.

### **Simplicial $n$ -Boundaries**

Again, similar to the singular homology case, a simplicial  $n$ -boundary is the boundary of a simplicial  $(n+1)$ -chain. A 2-boundary is the triangulated boundary of a solid 3-dimensional structure (a collection of consistently oriented 3-simplices). Similarly, a 1-boundary is the boundary of an orientable closed surface with boundary. Finally, we get that a 0-boundary is the boundary of a collection of directed line segments in your space, so a collection of points with an equal number of positively and negatively oriented points. Note however that if there are two points which do not have a path between them, then these two points are not a 0-boundary.

### **The $n$ -th Simplicial Homology Group**

As in the case of singular homology to get the  $n$ -th simplicial homology group (still denoted by  $H_n(X)$  because the simplicial and singular homology groups of a space are isomorphic), we take all the simplicial  $n$ -cycles. Then we quotient by the simplicial  $n$ -boundaries (again if  $x$  is a simplicial  $n$ -boundary, then we add the relation  $x = 0$ ). But now, we can express this in a much more concrete manner.

## 2.5 Examples

Now, that we have the powerful tool of simplicial homology at our disposal, we can get to calculating the homology groups of some real spaces.

### Homology of a Bouquet of $n$ Circles

Let  $X$  be the bouquet of  $n$  circles. Let's start by triangulating  $X$  (for our example, as in Figure 2.18, we will take  $n = 3$  but note that the argument will hold for any  $n$ ). We have just split up our space

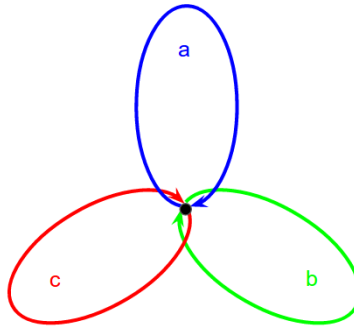


Figure 2.18: The bouquet of 3 circles with a given orientation on the edges. They are labeled as  $a$ ,  $b$ , and  $c$  and we will call the middle point  $p$ .

into a union of 1-simplices with 1 shared point in the center. Note that this agrees with our definition of a triangulation, we have 1-simplices which only meet each other at 0-simplices. Since there are no  $n$ -simplices for  $n \geq 2$  we can only look at the 1-chains and 0-chains and so we can only calculate  $H_1(X)$  and  $H_0(X)$ .

Let's first calculate  $H_0(X)$ . We know that 0-cycles and 0-chains are the same, so our initial cycle group is  $\langle p \rangle$ . Now, we have to quotient by all of the 0-boundaries. However, we know that any 0-boundary is a sum of elements of the form  $x - y$  where  $x$  and  $y$  are 0-simplices in your space. So, we get that all of the boundaries are a sum of  $p - p = 0$ . Therefore, we are only quotienting by 0 (that is, we only add the relation  $0 = 0$ ) which means that the group is unchanged. So, we can see that  $H_0(X) = \langle p \rangle$ , i.e. the free group generated by  $p$ . So,  $H_0(X)$  is isomorphic to the integers (denoted by saying  $H_0(X) \approx \mathbb{Z}$ ). Now, it is not immediately clear what a 1-dimensional hole might be, so let's work out a few more examples before we try to make an interpretation.

Now, for  $H_1(X)$  we have to do a little more work. However, we can take solace in the fact that we know  $H_1(X)$  is just the abelianization of  $\pi_1(X)$ , so we know that  $H_1(X) \approx \mathbb{Z}^3$  (or if we go back to the general case  $H_1(X) \approx \mathbb{Z}^n$ ). We can see that any 1-chain is of the form  $la + mb + nc$ , and so taking the



boundary, we can see that

$$\partial(la + mb + nc) = l\partial(a) + m\partial(b) + n\partial(c) = l(p - p) + m(p - p) + n(p - p) = 0$$

So, every 1-chain is a 1-cycle. Now, since there are no 2-simplices, then there are no 2-chains, and so there are no 1-boundaries. Therefore, we are again quotienting by 0 and so  $H_1(X)$  is equal to the free abelian group generated by  $a$ ,  $b$ , and  $c$  which is isomorphic to  $\mathbb{Z}^3$ . This meshes with our idea that the 1st homology group tells us how many 2-dimensional holes you have in your space, because just by looking, we can see that there are three holes there.

Now, these two homology groups were a good introduction, but they were relatively simple, because in both cases, we only quotiented by 0. Let's give ourselves more of a challenge with the next example.

## The 2-Sphere

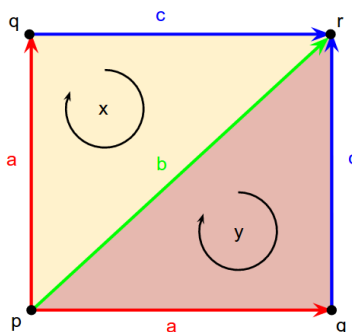


Figure 2.19: A way to construct the sphere by gluing up a square. The triangles are  $x$  and  $y$ , the edges are labeled as  $a$ ,  $b$ , and  $c$  and the points are  $p$ ,  $q$ , and  $r$ .

One way that we can triangulate the 2-sphere is by breaking it up into two triangles and sewing the triangles together (like how you might close a purse) as in figure 2.19. Now, for this, we are going to be choosing default orientations for the simplices and negative values of the simplex will denote copies of the simplex oriented in the other direction (for examples  $-2x$  denotes 2 copies of  $x$  with a counterclockwise orientation). Note that even though there are five lines drawn in the picture, there are actually only three 1-simplices. The 1-simplices denoted by  $c$  both represent the same 1-simplex, and similarly with both  $a$ 's and both  $q$ 's. We have had to make some compromises about the placement of our simplices so that the triangulation will fit within the 2-dimensional paper. For this to work, when actually constructing your object you must glue all copies of the same simplex together.

Let's first calculate  $H_0(X)$ . We know that all 0-chains are 0-cycles, so our initial cycle group is the free abelian group generated by  $p$ ,  $q$ , and  $r$ . Next, we need to quotient by all of the boundaries. We can

see that every boundary is some combination of the boundaries of  $a$ ,  $b$ , and  $c$  which are  $q - p$ ,  $r - p$  and  $r - q$ . So, if we add a relation for each of these, we get  $p = q$ ,  $q = r$ , and  $r = q$ . This means that we can convert  $p$ ,  $q$ , and  $r$  freely between each other, so given an arbitrary 0-cycle  $lp + mq + nr$ , we can convert them all to  $r$ , resulting in  $(l + m + n)r$ . There is a representation of an element in the homology group as copies of  $r$  and adding two different 1-cycles will just add the coefficients of  $r$ . This means that the free abelian group on one generator is a good candidate for the homology group. The only thing we have to worry about is whether or not any two distinct coefficients of  $r$  represent the same element of the homology group. However, this is not the case because that would mean there is some  $l$  and  $m$  such that  $lr = mr$  implying  $(l - m)r = 0$ , so the two points are indeed the same. Therefore, we have  $H_0(X) \approx \mathbb{Z}$  yet again. Hmm, perhaps a pattern is appearing...

Now, let's consider  $H_1(X)$ . We start out with all 1-cycles, which we can just find by hand (start at a point and then trace out a path which doesn't backtrack on itself and you will come up with surprisingly few options). We get that the only 1-cycles are  $n(a + c - b)$  for some  $n \in \mathbb{Z}$ . However, we can also see that  $n(a + c - b)$  is the boundary of  $n$  copies of  $x$ , so every 1-cycle is also a 1-boundary. If we quotient by the whole group, we just get the trivial group. This connects to what we already know for two reasons, the fundamental group of the 2-sphere is trivial so the abelianization of the trivial group is, again, the trivial group, and there are no 2-dimensional holes in the 2-sphere.

Now, if we look at  $H_2(X)$ , we can see that if you want a non-trivial 2-cycle, you have to start with  $n$  copies of either  $x$  or  $y$ . Let's assume we start with  $x$ . Then we can see the boundary is  $n(a - b + c)$ . But we want to cancel this out, and the only ways to do this are by either subtracting off  $n$  copies of  $x$  (leaving us with the trivial element) or by adding  $n$  copies of  $y$ . Therefore  $nx + ny$  is a 2-cycle. However, we can also see that we would get something similar if we started with  $ny$ . Therefore the cycle group consists of elements of the form  $nx + ny$ . However, there are no 2-boundaries, so the group is just the free abelian group generated by  $x + y$  (which is isomorphic to the integers). This fits with our intuition because the 2-sphere is the primary example of an object with a 3-dimensional hole.

## Two Points

Now, for reasons that will soon become clear, let us look at the space consisting of two distinct points.

Let us first begin by noting that only the 0th homology group exists here since there are no higher dimensional simplices. Also, the calculation of the 0-th homology is relatively easy. We know the free abelian group generated by  $p$  and  $q$  is the 0-cycle group and since there are no 0-boundaries to quotient by,  $H_0(X) = \mathbb{Z}^2$ .



Figure 2.20: The space of two distinct points,  $p$ , and  $q$ .

We did it! It's not  $\mathbb{Z}$ ! But what does this mean? We can see that this should mean that there are two 1-dimensional holes, but that still leaves us with two more questions, "What is a 1-dimensional hole?" and "how does this space have two of them?" It turns out that the interpretation of holes here breaks down a little bit. The 0-th homology group determines the number of *path-connected components* your space has. If a space is path-connected, that means that between any two points in your space, you can find a path whose endpoints are those points. So, a path-connected component of a space  $X$  is a maximal path-connected subset of  $X$  (by maximal, we mean that if you add any other points to your subset, the subset will no longer be path-connected). In all the previous examples there was only one path-connected component and that is why we had  $\mathbb{Z}$  as the 0th homology group.

Now, if we consider the real line, we can see why it might make sense that this could be related to 1-dimensional holes. If we take a "1-dimensional hole" out of the real line (if we subtract an interval), we see that we end up with two path-connected components. So, you can resolve this in several ways. You can say that every space starts out with a 1-dimensional hole and the 0th homology group counts the additional holes. You can also just consider the 0th homology group as a special case which does not in fact count holes, but just gives you some idea of how disconnected your space is. And finally, you can redefine the boundary of a 0-chain to be the number of points with regard to orientation of the points (a positive point is mapped to 1, a negative point is mapped to -1) and define what is called *reduced homology*. The author thinks the latter two methods are preferable, but so as not to cause confusion by redefining part of our homology theory, we will consider the 0th homology group as a special case.

## The Real Projective Plane

In the previous chapter, we introduced the real projective plane and mentioned some constructions of it. The first one we mentioned is by gluing opposite sides of a square together in opposite directions. By using this construction, we can find a triangulation of the real projective plane.

Now, we will skip the calculation of the 0th homology group as we can see that it is  $\mathbb{Z}$  because the real projective plane has 1 connected component and because the calculation of it is similar to the calculation for the sphere except slightly simpler.

So, we will start with the first homology group. By inspection there are two types of 1-cycles,

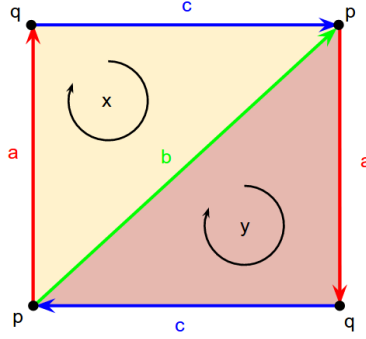


Figure 2.21: A gluing construction of the real projective plane  $\mathbb{R}P^2$ .

multiples of  $b$  and multiples of  $a + c$  (to see this pick a point and then trace out a path until you come back to your original point, you will see that these are indeed the only two types of paths). The boundaries are  $\partial(mx + ny) = m(a - b + c) + n(a + b + c)$ . So, if we choose  $m = -1$  and  $n = 1$ , we get that  $2b$  is the boundary of  $x - y$ . But then every element of the 1-cycle group can be expressed uniquely as  $m(a + c) + nb$ . Rearranging this,

$$m(a + c) + nb = ma + mc + nb = ma + mb + mc + nb - mb = m(a + b + c) + (n - m)b$$

So, by introducing some sort of change of variable here (or a change of basis), we can write each element uniquely as a linear combination of  $a + b + c$  and  $b$ . However, we just saw that we added the relation  $a + b + c = 0$ , so  $m(a + b + c) + nb = nb$ . And so we end up with the free group on  $b$  with the added relation  $2b = 0$ . With this additional relation, we can cancel out any even pairs of  $b$ 's to get that the only two elements of the group can be expressed by 0 and  $b$  with the operation that  $b + b = 0$ . We have actually seen a version of this group before in the calculation of the fundamental group for the real projective plane, which makes sense as this group is abelian, so its abelianization is itself. But what does this mean in terms of holes? Do we have a portion of a 2-dimensional hole? It turns out that the number of holes is really determined by the number of copies of  $\mathbb{Z}$  you have in the homology group, called the *rank* of the group. The other portion of the group, consisting of a finite group (or a group where every element can be added to itself enough times to be 0), is called the *torsion* of the group, and the torsion of the group tells you what types of unexpected and funky structure you have in your space, in this case that a surface is a closed non-orientable surface.

Now, the calculation of the 2nd homology group is relatively easy. Start by noticing that in the

2-chain group, every element can be expressed by  $mx + ny$ . Calculating its boundary, we get

$$\partial(mx + ny) = m(a - b + c) + n(a + b + c) = (m + n)a + (n - m)b + (m + n)c$$

So, in order for  $\partial(mx + ny) = 0$  to hold, we must have  $m + n = 0$  so that the coefficients on  $a$  and  $c$  vanish, giving us  $m = -n$ . We must also have  $n - m = 0$  so that the coefficient on  $b$  vanishes, giving us  $n = m$ . So, this then gives us  $n = m = -n$  and so  $n$  must be 0. Similarly, since  $n = m$ ,  $m$  must be 0. So, the only 2-cycle is 0. We do not even need to consider boundaries because the 2-cycle group is the trivial group. So, the real projective plane does not bound any 3-dimensional holes, a fact not immediately obvious from its construction.

## 2.6 Final Thoughts on the Homology Groups

Now, it might seem that the two types of homologies that we have talked about, singular and simplicial, are radically different, but thankfully it turns out that the resulting groups are isomorphic. However, the proof of this fact is quite long and complicated, so we direct you to [3] for a complete proof. While the homology groups are quite powerful and relatively easy to calculate, their power and ease are significantly augmented by the fact that there are extremely powerful tools that you can use to make calculation easier, some of which we will describe here in case the reader wants to explore them in more detail.

For example, there are tools which help you determine what would happen to the homology groups if you were to crush a portion of your space down to a point, tools determine what would happen if you were to glue together spaces, or tools to even just help calculate the homology groups, all in terms of homology groups which are much easier to calculate.

If you know the homology groups of a space and the homology groups of a subspace, then you can quite easily calculate the homology groups of the space resulting from crushing the subspace down to a point using a tool called *relative homology*. Similarly, if you have two intersecting spaces, you know the homology groups of the original spaces, and you know the homology groups of the intersection, you can quite easily calculate the homology groups of the union of the spaces, using *Mayer-Vietoris sequences*. This can be very effective in gluing constructions, when you know the homology of the groups of the original spaces and you are gluing the spaces together along a simple subspace of each of them.

These constructions use algebraic structures called *exact sequences* which provide tools for filling in gaps about long chains of inter-connected groups. It turns out that the homology groups themselves form an exact sequence which means that if you know most of the homology groups for a space, it can

make it a completely trivial exercise to calculate the remaining homology groups. Again, to learn more about these fascinating subjects and some of their applications, check out [3].

However, there are definitely some drawbacks to homology groups. The homology groups are much less powerful than the homotopy groups in most cases. There are plenty of spaces with isomorphic homology groups which can be easily distinguished by examining their homotopy groups. This occurs even just in the fundamental group. One common way to attempt to distinguish two knots is by subtracting the knot from  $\mathbb{R}^3$  and looking at the fundamental group of the space (called the *group of the knot*). However, it turns out that the first homology group of this space will always be  $\mathbb{Z}$ , so trying to distinguish knots using the first homology group of this space is quite futile (however, there are other spaces related to knots whose homology *can* distinguish knots!).

### 3 Cohomology: Oh, The Cochains that Bind Me!

In the previous section, we learned about a subject which is integral to algebraic topology, homology. There is a complementary subject, cohomology, which in essence is the same, but behaves remarkably differently. One major difference which often makes cohomology more suitable to work is that there is a (relatively) natural way to define a type of multiplication on cohomology elements called the *cup product* which augments the already-present addition. This turns the cohomology groups into an algebraic structure called a *graded ring*, a structure which behaves a lot like a ring of polynomials. However, this added structure comes at something of a cost. While the definition of homology and cohomology are quite similar, the actual subjects being studied in cohomology are significantly more abstract than those in homology, making it a more difficult subject. However, there are still some very well-founded interpretations to rely on in cohomology which we will make great use of. Unlike the section on homology, we will define cohomology rigorously first and then examine its interpretations.

#### 3.1 Definitions for Singular Cohomology

We will see very shortly that cohomology differs from homology in three main aspects.

- In cohomology, all the terms are the same as in homology except that you add the prefix co- to the beginning of all them (the *coaxiom*).
- In cohomology, maps go in the opposite direction than in homology (the *mapflip* axiom).
- Anything which originally used subscript notation switches to superscript notation (the *super-script* axiom).

These will be our three main axioms for converting homology to cohomology and will be referred to by their labels. So, let's start with the base concept, the definition of an  $n$ -cochain (a notation defined using the coaxiom). The reader should be warned that these are not serious axioms for a definition of cohomology, just a comment on how related these two subjects are.

### **n-Cochains**

An  $n$ -cochain is a labeling of each singular  $n$ -simplex in your space by an integer. For example, a 1-cochain is a way of assigning each path in your space an integer. More formally, an  $n$ -cochain is a function  $\phi$  from the set of singular  $n$ -simplices to the integers. We also have one added restriction, if  $x$  is a singular  $n$ -simplex and  $-x$  is the same simplex with opposite orientation, then  $\phi(-x) = -\phi(x)$ . With this added restriction, we can extend  $\phi$  to a linear function from the  $n$ -th chain group to the integers. So, if  $n$  and  $m$  are (not necessarily positive) integers and  $x$  and  $y$  are singular  $n$ -simplices, then on the chain  $nx + my$ , we have  $\phi(nx + my) = n\phi(x) + m\phi(y)$ .

We can see that there is a group structure on the set of  $n$ -cochains. The operation is addition of the outputs, so if  $\phi$  and  $\psi$  are cochains, then  $\psi + \phi$  should be a function from the singular  $n$ -simplices to the integers, so it makes sense to define it by what it does to each input. For an  $n$ -simplex  $x$ ,  $(\psi + \phi)(x) = \psi(x) + \phi(x)$ . For example, if  $f_1$  and  $f_2$  are the constant 1 and 2 functions respectively, then  $f_1 + f_2$  is the constant 3-function. Without too much work, we can show that the identity element is the 0-functions and the inverse of a cochain  $\phi$  is  $-\phi$ . Now, we need a notation for the group of cochains, so by applying the superscript axiom, we get that the group of  $n$ -cochains is denoted by  $C^n(X)$ .

Note that one of the ways that we looked at an  $n$ -chain was by taking a collection of simplices (possibly with repetition). Essentially, we were taking finitely many simplices and assigning an integer to them (after choosing a default orientation for each). For an  $n$ -cochain we are assigning an integer to *every* simplex simultaneously. This is one of the reasons why cohomology has such a significant gap in complexity over homology. It is often much easier to think about finite subsets of uncountably infinitely many things than to think about the entire uncountable set itself.

### **The Coboundary map**

An integral part of homology was the boundary map,  $\partial : C_n \rightarrow C_{n-1}$ . It made our definitions of cycles and boundaries possible, without which, there are absolutely no homology groups. We want some sort of analogue for cohomology, and by applying the coaxiom, we can deduce that it is called the coboundary map. Recall that by the mapflip axiom, maps go in the opposite direction and thus the coboundary map, denoted by  $\partial^*$ , is from  $C^n$  to  $C^{n+1}$ . Given an  $n$ -cochain  $\phi : C^n \rightarrow \mathbb{Z}$ , we can see

$\partial^*\phi$  must define some element which maps every singular  $n + 1$  simplex to an integer. For an  $n + 1$  chain,  $x$ , we define  $(\partial^*\phi)(x)$  to be  $\phi(\partial(x))$ . Unpacking this definition, in order to get the coboundary of a cochain  $\phi$  on a specific  $n + 1$  dimensional element, you must first take the boundary of that element, resulting in an  $n$  dimensional element. Then, since  $\partial(x)$  is a singular  $n$ -simplex, you can then apply  $\phi$  on it to give you the value of  $\partial^*\phi(x)$ . So, parallel to homology, an element of the cochain group is called  $n$ -coboundary if it is the coboundary of an  $(n - 1)$ -cochain.

After this point, it seems that everything is parallel to homology. We just need to define an  $n$ -cocycle and then the  $n$ -th cohomology group will be the  $n$ -cocycles mod the  $n$ -coboundaries.

### n-Cocycles

An  $n$ -cocycle is an  $n$ -cochain whose coboundary is the 0-function. While the statement is very simple, it does not immediately tell us how to recognize an  $n$ -cocycle or any of its properties. Let's start unpacking the definition. If  $\phi : C_n \rightarrow \mathbb{Z}$  is a cocycle, then  $\partial^*\phi(x) = 0$  for all  $x$  in  $C_{n+1}$ . Therefore

$$0 = \partial^*\phi(x) = \phi(\partial(x))$$

Therefore, the  $n$ -cocycles are exactly the functions which are 0 on all of the  $n$ -boundaries. One consequence of this is that if you can split up a singular  $n$ -simplex into multiple smaller simplices all with the same orientation, the value of the cochain on the large simplex is the sums of the values on the smaller simplices. Let's look at a few examples of why this might be true. Let  $\phi$  be a cocycle

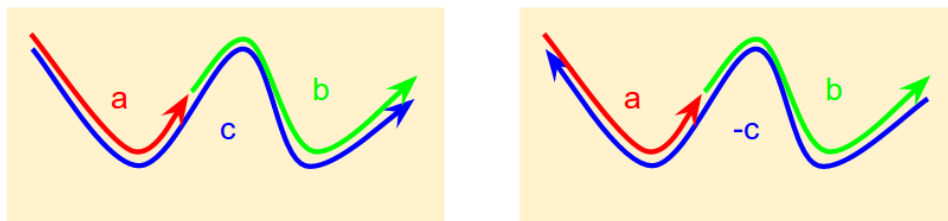


Figure 3.22: The singular 1-simplices  $c$  and  $-c$  together with  $a$  and  $b$

and suppose we wanted to break up some singular 1-simplex  $c$  into the simplices  $a$  and  $b$  (note that in Figure 3.22  $a$  and  $b$  actually lie inside  $c$ , but a picture with all of them on top of each other would be significantly less helpful than the current one). Then, we just need to show that  $\phi(a) + \phi(b) = \phi(c)$ . It is relatively clear from the figure that  $a$ ,  $b$ , and  $-c$  are the boundary of a very poorly embedded disk into our space (it got smushed in the process). Because we know that  $a + b - c$  is a boundary, we can



use the fact that  $\phi$  is a cocycle to deduce that  $\phi(a + b - c) = 0$ . Then we can further deduce

$$\phi(a + b - c) = 0 \implies \phi(a) + \phi(b) - \phi(c) = 0 \implies \phi(a) + \phi(b) = \phi(c)$$

A similar argument holds for 2-cochains. If we want to split up  $c$  into  $a$  and  $b$ , we can look at  $c$

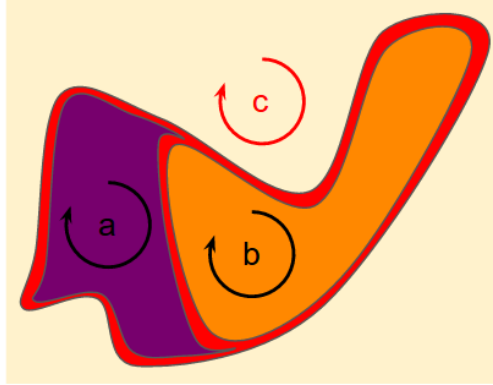


Figure 3.23: The singular 2-simplices  $c$  and  $-c$  together with  $a$  and  $b$

oriented in the opposite direction. Then, we consider  $c$  as the “other side” of a 3-ball which has been embedded in the space by smushing it down into a disk (or you can think of  $a$  and  $b$  as bulging ever so slightly out of the plane of the paper). Then from here on we can use the same argument as for the 1-dimensional case.

We can make another observation arguing in a similar manner as above. For a cocycle  $\phi$ , if we deform a simplex  $x$  to a simplex  $y$  without moving its boundary, then  $\phi(x) = \phi(y)$ . Let's start by noting that if we fix the boundary of an  $n$ -chain and move the interior, it is equivalent to moving the  $n$ -chain through a ball of dimension  $n + 1$ . For example, if we have a singular 1-simplex  $a$  and we fix its endpoints, when we move it, it sweeps out a disk where its final resting place is  $b$ . This way of moving  $a$  is called a *homotopy of  $a$  relative to its boundary* and we can say that  $a$  and  $b$  are homotopic to each other relative to their endpoints. From this, it is clear that  $a - b$  bounds a disk and so  $\phi(a - b) = 0$  and

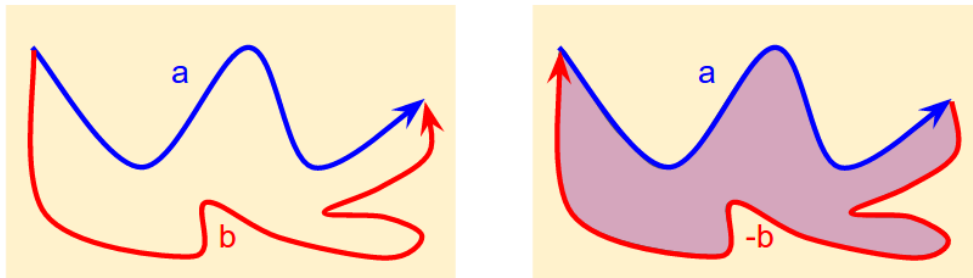


Figure 3.24: The singular 2-simplices  $c$  and  $-c$  together with  $a$  and  $b$

so  $\phi(a) = \phi(b)$ . We again can see that this type of argument will hold for higher dimensions, but due to the low-dimensional nature of the paper this is printed on, we will not include any visuals for this.

One final question we might have about cocycle concerns  $n$ -cocycles where there are no  $(n + 1)$ -chains. In this case, we say that all  $n$ -coboundaries are cocycles (by declaring that  $C_{n+1}$  is just the trivial group).

## n-Coboundaries

The definition of an  $n$ -coboundary is quite predictable. An  $n$ -coboundary  $\phi$  is an  $n$ -coboundary if it is the coboundary of some  $(n - 1)$ -coboundary  $\psi$ . That is,  $\phi$  is an  $n$ -coboundary if there exists some  $\psi$  such that  $\partial^*\psi = \phi$ . If we examine the definition more closely, for an  $n$ -chain  $x$

$$\phi(x) = (\partial^*\psi)(x) = \psi(\partial(x))$$

Knowing nothing about  $\psi$ , it would seem that this doesn't tell us much about  $\phi$ . However, we do get the incredible fact that for any singular  $n$ -simplex, the value of  $\phi$  is solely determined by the boundary of that simplex.

We can get a lot of nice properties of coboundaries out of this fact. First, we can deduce that an  $n$ -coboundary  $\phi$  (the coboundary of some  $\psi$ ) must also be an  $n$ -cocycle. We want to show  $\phi(x) = 0$  for some  $n$ -boundary  $x$ . If  $x$  is an  $n$ -boundary, then  $x = \partial y$  for some  $(n + 1)$ -chain  $y$ .

So,

$$\phi(x) = \phi(\partial(y)) = (\partial^*\psi)(\partial(y)) = \psi(\partial(\partial(y)))$$

We know from the section on homology that if you take the boundary of a boundary, then you get 0 (it's the reason that all boundaries are cycles). And it is not difficult to see that  $\psi(0) = 0$  since for any  $z$ , you get

$$\psi(0) = \psi(z - z) = \psi(z) - \psi(z) = 0.$$

Therefore, we get all the nice properties of cocycles for free.

However, one property of cocycles is pretty obvious for coboundaries; if you fix the endpoints and move a path, the value does not change. Actually, for a coboundary, we get something stronger than this. Consider the two paths in figure 3.25. They clearly have the same endpoints, but no matter how hard we try, we cannot move them together without moving their endpoints. So, if  $\phi$  is a coboundary,

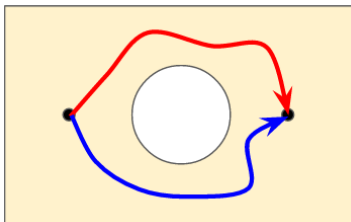


Figure 3.25: The red and blue paths share endpoints but are not homotopic relative to their endpoints.

its values on the red and blue paths must be equal, while this might not be the case if  $\phi$  is a cocycle which is not a coboundary.

A final note we have to make about coboundaries is the special case of a 0-coboundary. By our definition, a 0-coboundary would be the coboundary of a  $(-1)$ -chain, but since these do not exist, we define that the only 0-coboundary is the 0-cochain which sends every point (0-simplex) to 0.

## 3.2 Cocycle and Coboundary Examples

We will soon find out that the  $n$ th cohomology group is defined as the  $n$ -cocycles mod the  $n$ -coboundaries, however when we try and apply this definition to a space, we immediately run into a problem. What do an  $n$ -cocycle and  $n$ -coboundary even look like? We know some of their properties, but it can still be difficult to visualize them because of the complexities of the objects we are working with. So, let us look at some concrete examples. It is pretty clear that we cannot go through and assign each individual singular  $n$ -simplex a different number (that would require a list of them and there are uncountably many of them!) So, we must define this map based on some description in terms of the elements the cochain is assigning numbers to. For these examples, we are going to be using a space  $X$ , a disk with a hole punched out of it (like in figure 3.25).

### 1-Coboundary Example

Let's first examine a 1-coboundary in  $X$ . A 1-coboundary is just the coboundary of some 0-cochain, so let us choose an arbitrary cochain and take the coboundary of it. Note that there are no restrictions on the 0-cochain, so we can just choose our favorite way of assigning an integer to each point in  $X$ . We will do this in a striped manner as in figure 3.26. At every blue point, the 0-cochain  $\psi$  is 1 and at every red point it is -1. We then define the 1-cochain  $\phi$  to be the coboundary of this map. For the value of  $\phi$  on a singular 1-simplex, we just need to look at the endpoints. You take the value at the head and subtract it from the value at the tail. Looking at some simplices in this space as in figure 3.27, let's

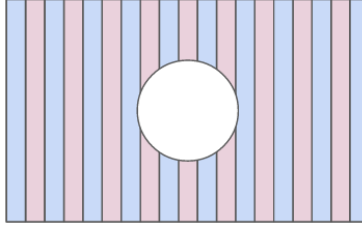


Figure 3.26: A 0-cochain  $\psi$  whose value is 1 on the blue points and -1 on the red points.

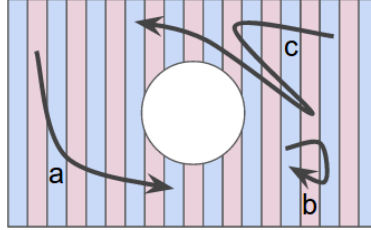


Figure 3.27: Three singular 1-simplices embedded in  $X$ .

find the value of  $\phi$  on them. Since the tip of  $a$  is in blue and its tail is in red,

$$\phi(a) = \psi(\text{blue}) - \psi(\text{red}) = 1 - (-1) = 2$$

Similarly,

$$\phi(b) = \psi(\text{blue}) - \psi(\text{blue}) = 0$$

$$\phi(c) = \psi(\text{blue}) - \psi(\text{blue}) = 0$$

$$\phi(-a) = \psi(\text{red}) - \psi(\text{blue}) = -2$$

Now, by using relatively simple 0-cochains, we can get a whole slew of 1-coboundaries. We could even try something really complicated, like defining the value of the 0-simplex to be the first non-zero digit in the ternary expansion of its distance from the center of the hole, but we will not go over any further cases of 1-boundaries (and the author implores you not to try and work out what the corresponding 1-coboundary would be in this case).

### 1-Cocycle Example

Now, if we instead want an example of a 1-cocycle (which is not a 1-coboundary), we have to think a little harder. For a 1-coboundary, we just needed to think of any 0-chain, and it immediately falls out. However, we now need to try and determine some way of assigning values to singular 1-simplices

which allow you to split up the simplices correctly but are not completely dependent on endpoints. We can start to think about this by looking at how a cochain can assign a value on a curve which closes up on itself. In this case, since the endpoint is the same as the beginning point, we know that the coboundary must be 0 on this simplex. However, a cocycle needs no such restriction as long as it is not the boundary of a disk. There is a clear example of such a loop in  $X$ , one which goes around the hole in the center and comes back to its start as in figure 3.28. So, let us declare that the value of our

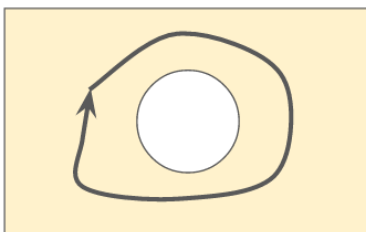


Figure 3.28: A closed curve which does not bound a disk in  $X$ .

cocycle  $\phi$  on this curve  $a$  is 1. If we now look at a curve that starts at the same point as  $a$  and wraps around the hole twice, we can perform a homotopy relative to its endpoints so that the point where it intersects itself is the endpoint. Then we can break the curve up into two curves and then perform a homotopy relative to the endpoints of the curves so that they lie in the same place. The value of  $\phi$  on this curve will have to be 2 because homotopy relative to endpoints does not affect the value of  $\phi$ . Then, extending this argument we arrive at the conclusion that for any loop based at that point (let

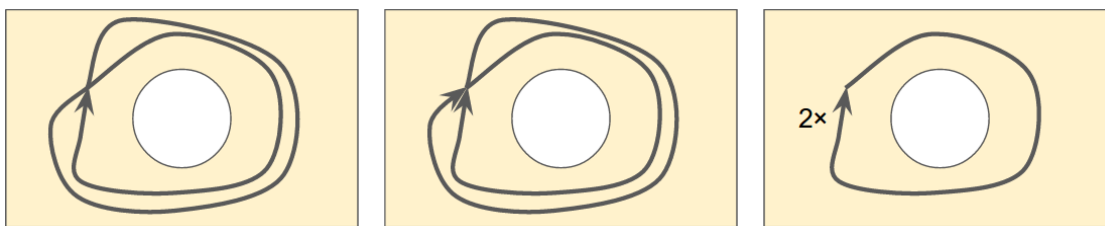


Figure 3.29: Going from twice a loop to a double-wrapped loop without changing the value of  $\phi$ .

us call it  $x$ ), the value of  $\phi$  only depends on the number of times it wraps around the hole. Also note that starting from a point  $y$  and then traveling to  $x$ , looping around the hole (possibly several times) to arrive back at  $x$  and then going back to  $y$  will also not change the value of the loop (as in figure 3.30) since we just add and subtract the value of  $\phi$  on the path between  $x$  and  $y$ .

We now know the value of  $\phi$  on any closed curve in  $X$ , but what about the curves which are not closed? One way is to define more rigorously what we mean by “wrapping around.” If we place a barrier from the outer boundary to the inner boundary in our space, we can then see if a loop wraps around

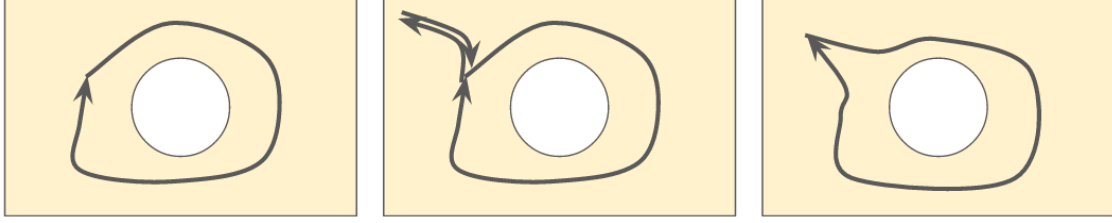


Figure 3.30: Starting from our original curve, we can add a path and its negative, and then perform a homotopy relative to that new basepoint to get a curve based at a different point with the same value in  $\phi$ .

the center hole, it must cross that barrier at least the number of times it wraps around the center. So,

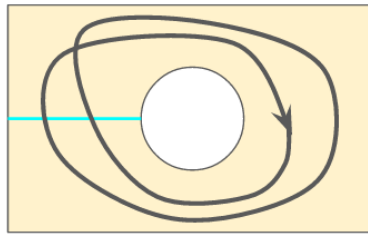


Figure 3.31: Our added barrier which intersects this double-wrapped curve twice.

we can naively start by saying that the number of intersections with the blue curve will determine the value of  $\phi$  on any curve in your space. However, we immediately see that this will not work because we know that  $\phi(-x) = -\phi(x)$  so we must have negative intersections.

We can then readjust our definition so that  $\phi$  counts the number of “signed” intersections of the curve with the barrier, where an intersection is positive if it goes through the barrier one way and negative if it goes through the barrier the other way. With this redefinition, we can see that this is

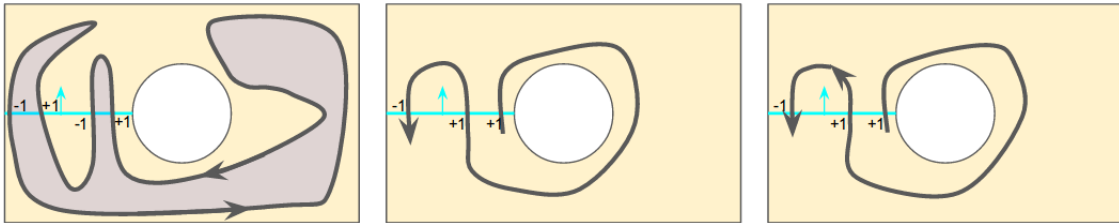


Figure 3.32: We can see that counting “signed intersections” works here.

indeed a cocycle which is not a coboundary. Any loop which bounds a disk will either not pass through the barrier or it will cross back and forth over the barrier so that all positive crossings will cancel with a negative crossing.

We started off with a couple of assumptions to get this cocycle. Firstly, this cocycle was not a coboundary. Secondly, this cocycle was non-zero on a specific closed curve. Thirdly, this cocycle was

exactly 1 on that specific closed curve.

If we wanted to modify our cocycle so that it had the first two properties but not the third, then we could initially define our cocycle to have a different value on that curve. We then arrive at the conclusion that we have a cocycle for each integer by assigning a different integer to that curve.

We can show that there is no cocycle which has the first property but not the second. In order for a 1-cocycle  $\psi$  to not be a 1-boundary, then there must be two paths  $a$  and  $b$  with the same endpoints for which  $\psi$  takes on two different values  $\alpha$  and  $\beta$ . Now, if we concatenate  $a$  with  $-b$  we get a curve which goes along  $a$  and then  $-b$  back to where it started. However,  $\psi(a - b) = \alpha - \beta \neq 0$ . From this we can gather that in order for a cocycle to not be a coboundary, it *must* take on a non-zero value on some closed curve. If we assume (correctly) that the only closed curves which do not bound disks are the ones which surround the central hole, then our cocycle is completely determined by its value on that closed curve and, given a cocycle, we can ask what its value is on the curve which surrounds the origin a single time. Then we can determine the entire cocycle, showing (very non-rigorously) that the family  $n\phi$  is the only family of non-coboundary cocycles.

### 3.3 The Definition of the $n$ th Cohomology group

Now that we have some idea of what a cocycle and coboundary might look like, let us define the  $n$ th cohomology group of a space. First, take all of the  $n$ -cocycles in your space. Then quotient by the  $n$ -coboundaries. Tada! We did it! We have defined the  $n$ th cohomology group, denoted by  $H^n(X)$  (using the superscript axiom). However, even with some ideas of what a coboundary and cocycle might look like, it can still be incredibly difficult to determine the cohomology groups due to the uncountably infinite nature of the objects involved. However, we can get some intuition behind them. If every  $n$ -cocycle was an  $n$ -coboundary then  $H^n(X)$  would be trivial. So, we are in essence looking for  $n$ -cocycles which are not  $n$ -coboundaries, and the more  $n$ -cocycles you have (or fewer  $n$ -coboundaries), the larger the group will end up being. We can think of the cohomology group as measuring your inability to find some  $\psi$  such that  $\phi$  is the coboundary of  $\psi$ . However, when things are uncountably infinite, it can be impossible for mere mortals (or computers) to find such a  $\psi$  even if it exists which is why we will jump straight to simplicial cohomology.

### 3.4 Simplicial Cohomology

Just like for homology, there is a way of reducing your uncountably infinite-dimensional groups down to finite-dimensional groups by using simplices. First we start by triangulating our space  $X$ .

Note that in order to make the examples easier, we have modified  $X$  slightly by making it round and we have “bent” our simplices in the triangulation to reduce their number. This object, a disk with a disk punched out of the middle is known as an “annulus.” This is the result of taking 4 triangles and then gluing them in a loop (as in the left of figure 3.33). The second picture is just a way to flatten the first after the gluing occurs.

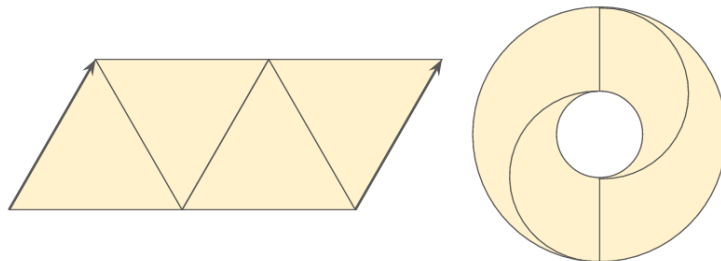


Figure 3.33: A triangulation of  $X$

Our  $n$ -cochains are ways of assigning an integer to each oriented  $n$ -simplex (where each of them has a default orientation). Since there are only finitely many of these  $n$ -simplices, we can sometimes just refer to them as a list of numbers. For example in the triangulation in 3.33 we could fix an orientation on each of the triangles and then each 2-cochain could be represented by a 4-tuple  $(w, x, y, z)$  where each of the letters corresponds to the value of the cochain on that specific triangle. Note that this is just  $\mathbb{Z}^4$  and treating it as such still does respect the group operation. If  $\phi$  and  $\varphi$  are two 2-cochains which have value  $a$  and  $b$  on a specific triangle, then adding  $\phi$  and  $\varphi$  will result in adding the corresponding entry in the 4-tuple. This way of representing a cochain will become very convenient when we get to the actual calculations of examples.

For a simplicial  $n$ -cochain  $\psi$  the simplicial coboundary map  $\partial^* \psi$  is a simplicial  $(n+1)$ -cochain  $\phi$  such that  $\phi(x) = \psi(\partial(x))$ . So, for example figure 3.34 shows a 0-cochain and its coboundary and 3.35 shows a simplicial 1-cochain and its coboundary. Now, it is quite trivial to check whether or not a simplicial cochain is a cocycle, just take its coboundary and see if all of the numbers are 0. For example, we can see that in 3.34 the coboundary is indeed a cocycle (a fact which we already knew, but it is still nice when our theorems match up with calculations). Now, a simplicial  $n$ -cocycle is an  $n$ -cochain whose coboundary is 0. As opposed to the singular cohomology case, we can actually determine which cochains are cocycles. We can see that an arbitrary  $n$ -cochain on  $X$  can be represented by a labeling of the  $n$ -simplices by arbitrary constants. Then if we want to determine which  $n$ -cochains are  $n$ -cocycles, take the coboundary and set the results to be 0. Then solve this system of integer equations (not an easy task, but doable if your triangulation is small enough).



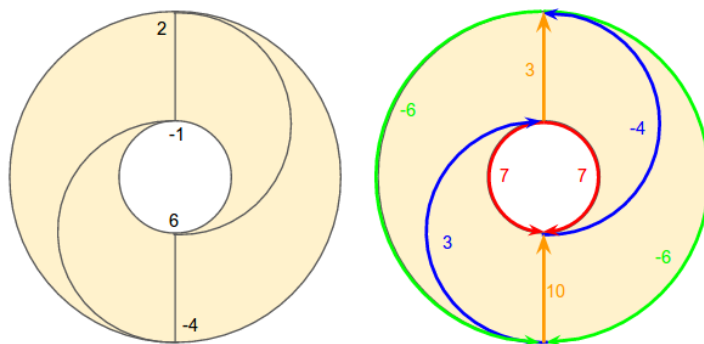


Figure 3.34: a 0-cochain (left) and its coboundary (right)

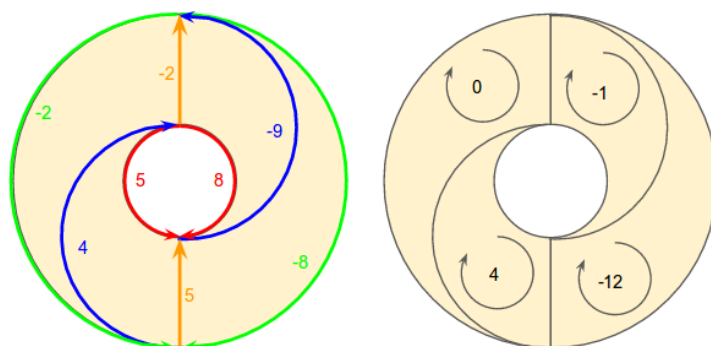


Figure 3.35: a 1-cochain (left) and its coboundary (right)

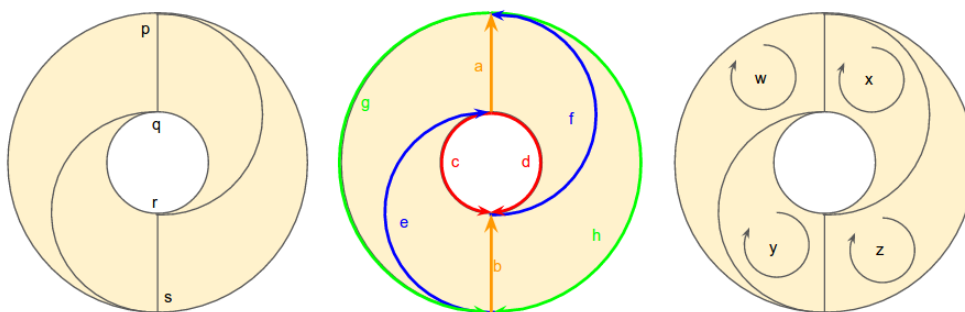


Figure 3.36: A way of representing arbitrary cochains in  $X$ .

For example, to determine which 1-chains are 1-cocycles, take the coboundary of an arbitrary 1-cochain (represented by the variables  $a$  through  $h$ ) and you get the following system of integer equations:

$$-e - g - a = 0 \quad a - f - d = 0 \quad a + b + f = 0 \quad e + c - b = 0$$

We could potentially solve this, but an integer programming problem in 8 variables with 4 constraints is a little too much to go through by hand, so we will wait and do this in the examples section (with an even further simplified triangulation of  $X$  using only 3 triangles!)

We can also similarly determine which  $n$ -cochains are  $n$ -coboundaries by taking the coboundary of an arbitrary  $(n - 1)$ -chain and seeing the form of the resulting cochain. So, to find the form of a 1-coboundary  $\phi$  in  $X$ , let us take the coboundary of an arbitrary 0-cochain and see the result:

$$\begin{array}{llll} \phi(a) = p - q & \phi(b) = r - s & \phi(c) = r - q & \phi(d) = r - q \\ \phi(e) = q - s & \phi(f) = p - r & \phi(g) = s - p & \phi(h) = s - p \end{array}$$

where  $p, q, r$ , and  $s$  are the values of an arbitrary 0-cochain on the four points. So, in order to determine if a 1-cochain is a 1-coboundary in  $X$  then you need to solve this integer programming problem in 12 variables with 8 constraints.

The above two examples show that even though simplicial cohomology is significantly more simple than singular cohomology, sometimes it can still be quite complicated. So let's do some examples to assuage these fears.

### 3.5 Simplicial Cohomology Examples

#### The Annulus

Let's again retriangulate the annulus but this time just with two triangles (as shown in figure 3.37). We can then represent all of the simplices of  $X$  as in figure 3.38.

So, let us first calculate the 0th cohomology group. We know that the only 0-coboundary sends  $p$  and  $q$  both to 0. Now, if a 0-cochain  $\phi$  is a 0-cocycle, it must satisfy the following four equations (due to the four 1-simplices)

$$\phi(p - p) = 0 \quad \phi(q - q) = 0 \quad \phi(q - p) = 0 \quad \phi(p - q) = 0$$

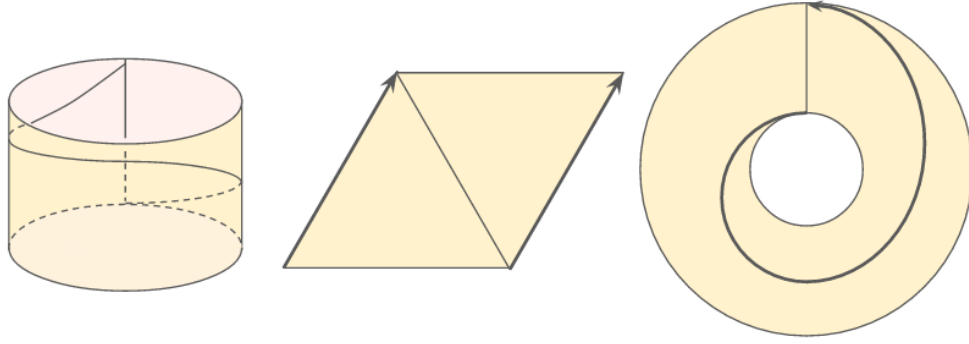


Figure 3.37: A triangulation of the annulus

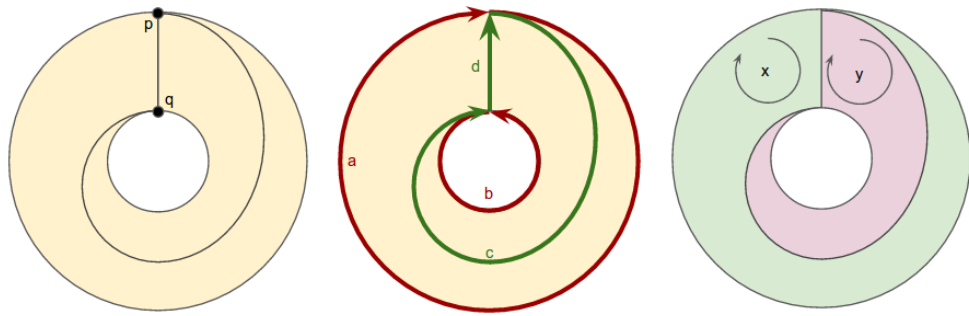


Figure 3.38: An arbitrary 0-cochain, 1-cochain, and 2-cochain

From these, we can easily see that the only restriction is that  $\phi(p) = \phi(q)$ . So, using pairs of numbers to represent the 0-cochains, we can see all of the 0-cocycles are of the form  $(n, n)$ . Then since we are only quotienting by  $(0, 0)$ , we get that the group is  $(n, n)$  which is isomorphic to  $\mathbb{Z}$  by rewriting  $(n, n)$  as  $n$ .

For the 1st cohomology group, let us start by clarifying that if  $\phi$  is a 1-cochain then we will represent it by  $(\phi(a), \phi(b), \phi(c), \phi(d))$  (as opposed to some  $(\phi(b), \phi(c), \phi(a), \phi(d))$  or some other permutation). So, first if we let  $\phi$  be a cocycle, then we see

$$0 = \phi(a) - \phi(d) - \phi(c) \quad 0 = \phi(b) + \phi(c) + \phi(d)$$

So,  $\phi(a) = \phi(d) + \phi(c)$  and  $\phi(b) = -\phi(c) - \phi(d)$ . Therefore, using our tuple notation, the 1-cocycles are exactly the 1-cochains of the form  $(m + n, -m - n, m, n)$ . Now, letting  $\phi$  be a coboundary,

$$\phi(a) = p - p \quad \phi(b) = q - q \quad \phi(c) = q - p \quad \phi(d) = p - q$$

where, in an abuse of notation,  $p$  and  $q$  are arbitrary values representing the value of some 0-cochain

on the points  $p$  and  $q$ . Translating this into tuple-notation, all of the 1-coboundaries are of the form  $(0, 0, q - p, p - q)$ . But any value  $p - q$  can be represented by another variable  $l$ , so we can reduce the above to  $(0, 0, l, -l)$ . Now, when we quotient by the coboundaries we can add the tuple  $(0, 0, n, -n)$  to our canonical representation of a cocycle,  $(m + n, -(m + n), m, n)$  in order to arrive at  $(m + n, -(m + n), m + n, 0)$ . However again  $m + n$  can be represented by some variable  $l$  so we get that every element of the cohomology group is representable by  $(l, -l, l, 0)$ . We again see that this is isomorphic to the integers by replacing  $(l, -l, l, 0)$  with  $l$ .

For the second cohomology group, we can see that since there are no 3-simplices, every 2-cochain is a 2-cocycle. So now let's determine the 2-coboundaries. For an arbitrary 2-cochain  $\phi$  we get the constraints

$$\phi(x) = a - d - c \quad \phi(y) = b + c + d$$

So, every 2-coboundary can be represented by  $(a - (c + d), b + c + d)$ . However, it is clear that every tuple can be represented this way by letting  $c$  and  $d$  be 0. Therefore, we are quotienting by everything in the group and so we are left with the trivial group.

So, if  $X$  is the annulus,  $H^0(X) \approx \mathbb{Z}$ ,  $H^1(X) \approx \mathbb{Z}$ ,  $H^2(X) \approx \{0\}$ .

Now, going back to our example of a cocycle which is not a coboundary in the annulus, we see that the first cohomology group is isomorphic to the integers, and there we found a family of cocycles which were not coboundaries which formed a group isomorphic to the integers. So, we could represent the elements of the cohomology group there by barriers with a particular value on them. We will revisit this interpretation of the 1st cohomology group when we discuss Poincaré duality, but for now, keep it in the back of your mind.

## The Bouquet of $n$ Circles

As in the homology example, we will again consider the case when  $n = 3$ . Let us start with the 0-th cohomology group. We know that there are no 0-coboundaries, so let us look at the 0-cocycles. We can immediately see that no matter what  $\phi(p)$  is, taking the coboundary of  $\phi$  on any of the 1-chains will give us  $\phi(p) - \phi(p) = 0$ . Therefore any 0-cochain is a 0-cocycle. So we get a copy of the integers generated by the 1-tuple  $(1)$  and we quotient by the trivial group, resulting in a group isomorphic to the integers.

For the 1st homology group, we know that all of the 1-cochains are 1-cocycles since there are no 2-simplices. Then determining the coboundaries, we can see that if  $\psi$  is a 0-cochain and  $x$  is a 1-chain,

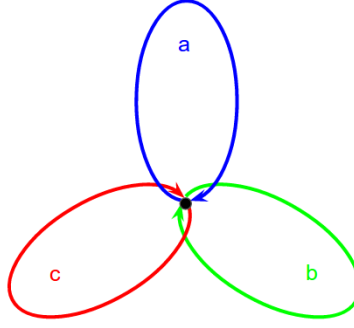


Figure 3.39: The bouquet of 3 circles with a given orientation on the edges. They are labeled as  $a$ ,  $b$ , and  $c$  and we will call the middle point  $p$ .

then

$$(\partial^* \phi)(x) = \phi(\partial(x)) = \phi(p - p) = 0$$

and so every coboundary is 0 on every simplex. So, we can see that we start out with  $\mathbb{Z}^n$  and then quotient by 0, leaving us with  $\mathbb{Z}^n$ .

## Two Points

If we again consider the space consisting of two discrete points,  $p$  and  $q$ , we can see that only the 0th cohomology group is relevant since there are no  $n$ -cochains for  $n \geq 1$ . Now, since there are no 1-simplices, the group of cocycles is just  $\mathbb{Z}^2$ . We also know that the only 0-coboundary is just  $(0, 0)$ . Therefore  $\mathbb{Z}^2 \text{ mod } (0, 0)$  results in  $H^0(X) \approx \mathbb{Z}^2$ .

This is similar to the 0th homology group for this space and so we might start wondering if the 0th homology and cohomology groups are indeed related. If we examine the 0th cohomology group of a general space  $X$ , we start out with all of the 0-cocycles. So, let  $\phi$  be a 0-cocycle. For any path  $a$  in  $X$  from  $p$  to  $q$ , we get that that

$$0 = (\partial^* \phi)(a) = \phi(\partial(a)) = \phi(q - p) = \phi(q) - \phi(p)$$

From this, we arrive at the conclusion that  $\phi(p) = \phi(q)$ . So, since  $a$  was general, we can see that  $\phi(x) = \phi(y)$  for any two points which have a path between them. That is,  $\phi$  is constant on all path-connected components of  $X$ . We also know that there are no 0-coboundaries, so  $\phi$  is determined completely by a choice of value on each path-connected component. Therefore, we can see that  $H^0(X) \approx \mathbb{Z}^n$  where  $n$  is the number of path-connected components of  $X$ .

Now, so far we have only seen examples where  $H^n(X) \approx H_n(X)$ . Unfortunately because most of our spaces discussed here are relatively sane (they do not do any weird twisting) their homology and cohomology groups are isomorphic (due to the Universal Coefficient Theorem which we will talk about later in the chapter). However, we do know of one space which does some weird twisting, which should give us some variety.

## The Real Projective Plane

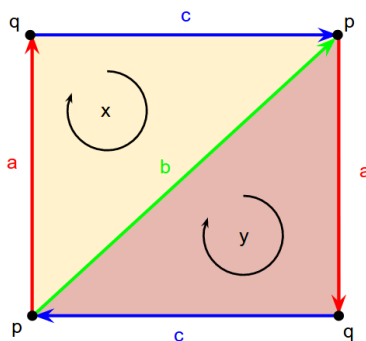


Figure 3.40: A gluing construction and triangulation of the real projective plane  $\mathbb{R}P^2$ .

Let us recall the triangulation of the real projective plane which we used to calculate the homology groups (figure 3.40). We know that since the real projective plane is path-connected,  $H^0(X) = \mathbb{Z}$ .

Now, for the first homology group, let us start with the 1-cocycles. For a 1-cocycle  $\phi$ , we get the following restrictions:

$$\phi(a + c - b) = 0 \quad \phi(a + b + c) = 0$$

So, we can see that  $\phi(b) = \phi(a) + \phi(c)$  from the first constraint and  $\phi(b) = -\phi(a) - \phi(c)$  from the second. However, if we multiply both sides of the second constraint by -1, we get

$$\phi(b) = \phi(a) + \phi(c) = -\phi(b)$$

So  $\phi(b) = 0$ . As a side-effect of this we can plug in 0 for  $\phi(b)$  in either constraint to get  $\phi(a) = -\phi(c)$ . Using the tuple-notation, we get that the 1-cocycles are of the form  $(n, 0, -n)$ . Now, looking for the 1-coboundaries, we can see that if  $\phi$  is an arbitrary coboundary, we get the following constraints:

$$\phi(a) = q - p \quad \phi(b) = p - p \quad \phi(c) = p - q$$

where  $q$  and  $p$  are the values of  $q$  and  $p$  on a 0-cochain. If we multiply the third constraint by  $-1$ , then we get  $\phi(a) = q - p = -\phi(c)$  and we get that the coboundaries are of the form  $(n, 0, -n)$ . So, we can see that quotienting all of the 1-cocycles by the 1-coboundaries gives us the trivial group.

What does this mean then? We know from an earlier discussion that a cocycle which is not a coboundary must take a non-zero value on some closed curve. However, we also know that a 1-cocycle must take the same value on two closed curves which are homotopic to each other. But we know that up to homotopy, there are only two curves, the constant curve and another curve, let us call it  $a$ . We know that  $\phi$  must take the value 0 on the constant curve since it bounds a (very poorly mapped) disk. So  $\phi(a)$  must be  $n$  where  $n \neq 0$ . However, we then get that

$$2n = 2\phi(a) = \phi(2a) = \phi(\text{constant curve}) = 0$$

which brings us to a contradiction. So, it does make sense that the first homology group is trivial since  $\phi$  being a cocycle implies that  $\phi$  is 0 on all closed curves.

Now, the second cohomology group is where things get interesting. Since there are no 3-simplices, all 2-cochains are 2-cocycles. Then looking at the constraints for a 2-coboundary  $\phi$ , we get

$$\phi(x) = a + c - b \quad \phi(y) = a + b + c$$

Denoting  $a + c$  by  $l$ , we can see that all the 2-coboundaries are of the form  $(l - b, l + b)$ . However, we can introduce a change of variables and see that this can also be written as  $(k, k + 2b)$  by letting  $k = l - b$ . On the other hand, every cocycle is of the form  $(m, n)$ . However, after quotienting, we can add  $(-m, -m)$  to get that every cohomology element is of the form  $(0, n - m)$ . We can then further reduce this by subtracting out  $(0, 2)$  until we get either  $(0, 1)$  or  $(0, 0)$ . We cannot reduce this further because that would require us to be able to write  $(0, 1)$  as  $(k, k + 2b)$  which is impossible (you need  $k$  to be 0 and then once you have that you must write 1 as  $2b$  for some integer  $b$ ). So, this group is isomorphic to the only group with two elements (denoted by  $\mathbb{Z}/2\mathbb{Z}$  (which we saw in the first homology group of the real projective plane and also its fundamental group).

Our last explicit cohomology calculation will bring us back into more visualizable territory.

## The Torus

We can create the torus by gluing opposite edges of a square together, so we can get a visualization as in figure 3.41. We know that the 0th cohomology will be isomorphic to  $\mathbb{Z}$  since it is path-connected.

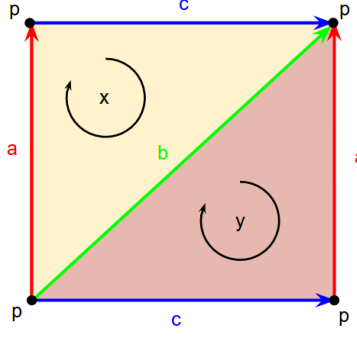


Figure 3.41: A gluing construction and triangulation of the torus.

For the first cohomology group, we can start by finding out the 1-cochains. If  $\phi$  is a 1-cochain then we get two constraints

$$\phi(a + c - b) = 0 \quad \phi(-a - c + b) = 0$$

However, we only really get one constraint here since they both imply  $\phi(a) + \phi(c) = \phi(b)$ . So all 1-cocycles are of the form  $(a, c, a + c)$ . Now for the coboundaries, since all of the 1-simplices are loops and coboundaries are 0 on loops, all of the coboundaries are 0 everywhere (just follow the argument for the bouquet of  $n$  circles). Therefore, we quotient only by 0. However, this cocycle is completely determined by the values of  $a$  and  $c$  to the point where we could just replace  $(a, c, a + c)$  with  $(a, c)$  and the resulting group would have the same structure (if you have some background in linear algebra, we can find something of a basis with  $(a, 0, a)$  and  $(0, c, c)$ ). Therefore this group is isomorphic to  $\mathbb{Z}^2$ .

For the 2nd homology group, we can start by noting that all 2-cochains are 2-cocycles since there are no 3-simplices. Then assuming  $\phi$  is a coboundary, we get the following constraints:

$$\phi(x) = a + c - b \quad \phi(y) = -a - c + b$$

So all of the coboundaries are of the form  $(a + c - b, -(a + c - b))$  and replacing  $a + c - b$  with  $d$  we get that the coboundaries are the cocycles which have the form  $(d, -d)$ . So, if we start with  $(m, n)$ , we can rewrite this as  $(m, n + m - m)$  and then we can replace  $n + m$  with  $l$  to get that all 2-cocycles can be uniquely written as  $(m, l - m)$ . Now, we have quotiented by all elements of the form  $(d, -d)$ , so we can subtract off  $(m, -m)$  from the representation of the 2-cocycle to get that they can be written as  $(0, l)$  which is isomorphic to the integers.



### 3.6 The Universal Coefficient Theorem

After calculating all of those cohomology groups, we saw that in a lot of the cases the homology groups and cohomology groups were actually isomorphic to each other. These spaces all were relatively well-behaved and were pretty easy to visualize. It wasn't until we got to the real projective plane that we found some disparities between the homology and cohomology groups, and even then, while they were different, they still both involved  $\mathbb{Z}/2\mathbb{Z}$ . This connection is very real and quite easily stated in the powerful *universal coefficient theorem*.

Before we fully state it, let us remind ourselves what rank and torsion are for an abelian group. Every abelian group  $G$  (that is finitely generated) has 2 components, some number of copies of  $\mathbb{Z}$  and some finite group  $T$ . The number of copies of  $\mathbb{Z}$  is called the *rank* and the finite group  $T$  is called the torsion subgroup. So, every finitely generated abelian group  $G$  has two components  $T$  and  $\mathbb{Z}^r$  for some  $r \geq 0$  and we denote this by  $G = T \oplus \mathbb{Z}^r$ , where the operation  $\oplus$  is called the *direct sum*.

Let us first go over the definition and some examples of direct sums.

#### Direct Sums

If  $A$  and  $B$  are two abelian groups with operations  $\circ$  and  $*$  respectively, then we can make a new group out of these called the *direct sum* of  $A$  and  $B$ , denoted  $A \oplus B$ . The elements of  $A \oplus B$  are pairs where the first item in the pair is from  $A$  and the second item in the pair is from  $B$ . In order to multiply two elements  $(a_1, b_1)$  and  $(a_2, b_2)$  in  $A \oplus B$  we define  $(a_1, b_1) \cdot (a_2, b_2) = (a_1 \circ a_2, b_1 * b_2)$ .

For example, one way to represent  $\mathbb{Z}/2\mathbb{Z}$  is the set  $\{1, a\}$  where  $1$  is the identity element and  $a^2 = 1$ . So, two elements from  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$  are  $(1, n)$  and  $(a, m)$ . Then we see that

$$(1, n) \cdot (a, m) = (1a, n + m) = (a, n + m)$$

#### Statement of the Theorem

Now, knowing direct sums we can state the theorem.

**Theorem.** *If the homology groups  $H_n(X)$  and  $H_{n-1}(X)$  of a space  $X$  have finite rank then the cohomology group  $H^n(X) \approx T \oplus \mathbb{Z}^r$  where  $T$  is the torsion subgroup of  $H_{n-1}(X)$  and  $r$  is the rank of  $H_n(X)$ .*

Unpacking the definition, we can see this says that if your space is nice enough then you can easily find the  $n$ th cohomology group from the homology group. You take the copies of  $\mathbb{Z}$  from the corresponding homology group and you take the torsion from the homology group one level down.

One immediate corollary of this is that a space's cohomology groups are completely determined by the homology groups (this turns out to be true even if your space is not very nice, but the statement of this is much more difficult). Another incredible fact is that if your homology groups have no torsion, then the  $n$ th cohomology group is isomorphic to the  $n$ th homology group. We can check this is true for a couple of spaces. For the bouquet of  $n$  circles, we saw that  $H_0(X) \approx \mathbb{Z}$  and  $H_1(X) \approx \mathbb{Z}^n$ . So, since there is no torsion here, we get that  $H^0(X) \approx \mathbb{Z}$  and  $H^1(X) \approx \mathbb{Z}^n$  as we previously calculated. We also saw that this was the case for the space consisting of two distinct points.

For a more difficult example, we can check our calculation of the homology and cohomology of the real projective plane. We calculated the homology groups as  $H_0(\mathbb{R}P^2) \approx \mathbb{Z}$ ,  $H_1(\mathbb{R}P^2) \approx \mathbb{Z}/2\mathbb{Z}$ ,  $H_2(\mathbb{R}P^2) \approx \{0\}$ . So, from this, we can calculate the cohomology groups. But before, we do, let us just make a note that any nonsensical homology or cohomology group (like ones with negative indices or ones whose indices are higher than the largest-dimensional simplex) we declare to be trivial in order for our calculations to make sense. We know that  $H^0(\mathbb{R}P^2)$  is the torsion of  $H_{-1}(\mathbb{R}P^2)$  (which is trivial by the previous sentence) direct sum with the non-torsion part of  $H_0(\mathbb{R}P^2)$  which is  $\mathbb{Z}$ . Therefore  $H^0(\mathbb{R}P^2) \approx \mathbb{Z}$ . For  $H^1(\mathbb{R}P^2)$  we know that  $H_0(\mathbb{R}P^2)$  has no torsion so  $H^1(\mathbb{R}P^2)$  is isomorphic to the non-torsion part of  $H_1(\mathbb{R}P^2)$ . However,  $H_1(\mathbb{R}P^2) \approx \mathbb{Z}/2\mathbb{Z} \approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^0$  and so  $H^1(\mathbb{R}P^2)$  has rank 0 and no torsion and so is trivial. For  $H^2(\mathbb{R}P^2)$ , we can see that  $H_2(\mathbb{R}P^2)$  has rank 0 (it is the trivial group) and  $H_1(\mathbb{R}P^2)$  has torsion  $\mathbb{Z}/2\mathbb{Z}$  and so  $H^2(\mathbb{R}P^2) \approx \mathbb{Z}/2\mathbb{Z}$ . We can see that these do indeed match up with our original hand-calculation of the cohomology groups, but this way was much less painful.

We can also now calculate the cohomology groups for the 2-sphere, having previously calculated the homology groups. We can see that there is no torsion, so the cohomology groups are isomorphic to the homology groups, namely  $H^0(S^2) \approx \mathbb{Z}$ ,  $H^1(S^2) \approx \{0\}$ , and  $H^2(S^2) \approx \mathbb{Z}$ .

For a bit more of a challenge that we do not know the answer to, let us calculate the cohomology of the Klein bottle,  $K$  just from its homology groups. Its homology groups are  $H_0(K) \approx \mathbb{Z}$ ,  $H_1(K) \approx \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ ,  $H_2(K) \approx \{0\}$ . Then we can see  $H^0(K) \approx \mathbb{Z}$  since  $H_0(K) \approx \mathbb{Z}$  and the torsion of the negative 1st subgroup is trivial. We can also see that for  $H^1(K)$ , the rank of  $H_1(K)$  is 1 and the torsion of  $H_0(K)$  is trivial so  $H^1(K) \approx \mathbb{Z}$ . And finally, for  $H^2(K)$ , the rank of  $H_2(K)$  is 0, but the torsion of  $H_1(K)$  is  $\mathbb{Z}/2\mathbb{Z}$  and so  $H^2(K) \approx \mathbb{Z}/2\mathbb{Z}$ .

### 3.7 Poincaré Duality

Let's examine the homology and cohomology of the torus. We can see that since there is no weird twisting that is going on, the homology and cohomology groups of the torus (denoted here by  $T^2$ ) are

isomorphic, therefore  $H_0(T^2) \approx \mathbb{Z}$ ,  $H_1(T^2) \approx \mathbb{Z}^2$ , and  $H_2(T^2) \approx \mathbb{Z}$  (this is not a rigorous calculation of the homology groups, but the calculations of them are not too difficult if the reader is unsatisfied). So, now let us try and come up with the actual elements of the first cohomology group using our system of “signed barriers” as in the annulus example. We can see that if we want to set up a barrier in the torus, we must use a closed loop like in figure 3.42.

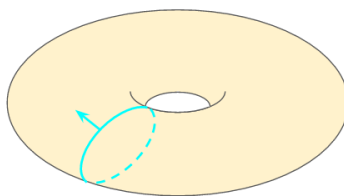


Figure 3.42: A signed barrier on the torus

Now, we know that if we have two loops which are homotopic to each other and have opposite orientations, then these two loops cancel each other in homology because they are the boundary of some 2-chain (consisting of one of the annuli which they bound). Similarly, in cohomology, if we have two signed barriers which have opposite orientation and are homotopic to each other then the resulting 1-cochain is actually a coboundary. Let’s see an example of this. If we look at figure 3.43, on the left we have two signed barriers which are homotopic to each other but with opposite orientations. Then on the right we can find a coboundary which is equivalent to this by assigning every point in between the curves on the left-hand side of the torus to be 1 and the remaining points to be 0. To see why these two are equivalent start at a point and then draw out a path going around the center hole. If the path doubles back on itself over one of the curves then the barrier will produce a  $+1$  and  $-1$  which will cancel. If the path makes a full traversal around the center hole then the values of the two separate barriers will cancel each other out. So all that remains to determine is if the path ends in the region it starts or the other one and so the path does not depend on its endpoints.

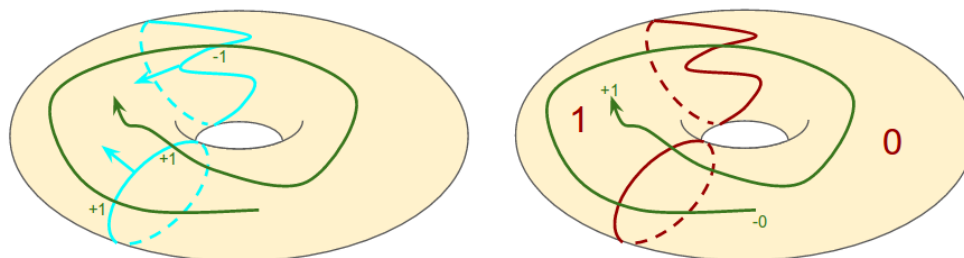


Figure 3.43: A signed barrier on the torus

Now, this is not the only barrier we could have drawn, we could also draw one going around the

middle hole which will similarly count the number of times a curve goes around the handle of the donut. However, we can see that by taking multiple copies of these two signed barriers, we can get

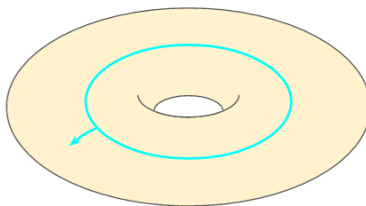


Figure 3.44: A different signed barrier on the torus

$\mathbb{Z}^2$  which is the cohomology group of the torus. Does this look kind of familiar? We have determined that the 1st cohomology group of the torus is represented by closed curves in the torus with a special equivalence relation: two curves are the same if they have opposite orientations and bound a closed region. If you said yes, it is because this is exactly the definition of the first homology group.

This is our first example of *Poincaré duality*. Poincaré duality says that if your space is nice enough, then there are some more nice relationships between certain homology and cohomology groups in addition to the ones stated by the universal coefficient theorem.

But before we can state the theorem, we have to go over one definition.

## Manifolds

An  $n$ -manifold is a topological space where every point is contained in an area which is *homeomorphic* to an  $n$ -ball. In order to exclude some strange and exceptional cases, we also need to include the restriction that the topological space can be embedded in some  $\mathbb{R}^m$  (where  $m$  might have to be significantly greater than  $n$ ). Two spaces are homeomorphic if there is a continuous invertible function whose inverse is also continuous between the spaces. However, you can also think of an  $n$ -manifold as a topological space where around every point there is an area which “looks like”  $\mathbb{R}^n$  (where “looks like” is in a topological sense, two spaces look alike if they can be stretched or bent so that they are the same). Let’s look at some examples of manifolds. A 1-manifold is an object where at every point, you can find an area which “looks like” the real line. Well, a circle or a bunch of circles fits the definition. Around every point, there is a circle arc which resembles a portion of the real line.

2-manifolds get a little more interesting. We can think of a couple right right off the bat. The 2-sphere works because at every point you can find a small curved disk which resembles a disk in the real plane. Similarly the torus,  $T^2$  is a 2-manifold for the same reason.

Now, a trivial example of a 3-manifold is  $\mathbb{R}^3$ , but let’s try and find one which is less obvious. Let

us start by considering a solid (filled-in) cube. The cube has six faces, front, back, top, bottom, right, and left. Now we are going to glue opposite faces together. The resulting object, known as the 3-torus

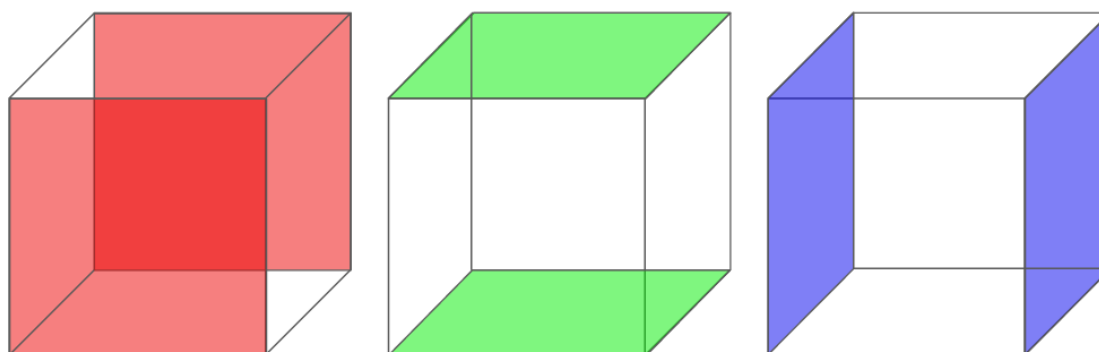


Figure 3.45: We glue together faces of the cube which have the same color

(denoted by  $T^3$ ) is a 3-manifold. If you do not try and physically perform the gluing by stretching the cube out, then this space is not too difficult to think about. Imagine that you are in a room, but the walls are such that whenever you try and walk into a wall, you just end up on the other side of the room. it helps to visualize this if you are currently in a room shaped like a rectangular prism. Looking at the wall in front of you, if you are standing facing a wall, if you were to walk into the wall, you would end up behind where you currently are. While the idea of this cube is relatively simple, you can end up with some very non-intuitive results. For example, If we stretch a square parallel to one of the faces in the 3-torus, we can see that opposite edges of that square will be identified, giving us a copy of the 2-torus.

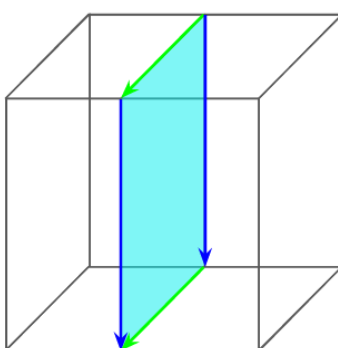


Figure 3.46: An embedding of the 2-torus in the 3-torus

You can also obtain different 3-manifolds by gluing opposite faces together with twists, similar to the way obtained the real projective plane by gluing opposite sides with different orientations, however the 3-torus is the only 3-manifold we will be talking about.

## Statement of the Theorem

**Theorem** (Poincaré Duality). *For an  $n$ -manifold  $M$  which is compact and orientable, then  $H^k(M) \approx H_{m-k}(M)$ .*

Now, we will only be dealing with manifolds up to dimension 3, so in this context a manifold is compact if you can find a triangulation of it which involves finitely many simplices (compactness is a way of ensuring that your space is somehow “small” or finite). Now, to determine whether or not an  $n$ -manifold is oriented, embed an oriented  $n$ -simplex into your space and if you can move the simplex through your space and back to the original point so that its orientation is different from when you started, your space is non-orientable.

Let us apply this to some manifolds that we know about. For a 1-manifold  $M$ , this tells us that  $H_0(M) \approx H^1(M)$  and  $H_1(M) \approx H^0(M)$ . However, we know that  $H_0(M)$  and  $H_1(M)$  both count the number of connected components, so this tells us that

$$\mathbb{Z}^c \approx H_0(M) \approx H^0(M) \approx H_1(M) \approx H^1(M)$$

where  $c$  is the number of connected components of  $M$ .

For a 2-manifolds  $M$  (a surface), we again know that what the 0th homology and cohomology groups do, so this tells us that  $H^2(M) \approx H_2(M) \approx \mathbb{Z}^c$  where  $c$  is the number of connected components. However, it also tells us that  $H_1(M) \approx H^1(M)$ . We saw an example of this in the 2-torus, every cohomology element corresponded precisely to a homology element.

The first case where things get really interesting is for 3-manifolds. If  $M$  is a 3-manifold, Poincaré duality tells us that  $H_1(M) \approx H^2(M)$  and  $H_2(M) \approx H^1(M)$ . So, if we combine this with the universal coefficient theorem, if the 3-manifold has no torsion in its homology groups, the 1st and 2nd homology and cohomology groups are all isomorphic. This is very nice because if we just want the rank of the groups, we only need to calculate one out of four of the 1st and 2nd homology and cohomology groups and we instantly know the rank of the other three.

But let's see an example of what Poincaré duality really means by using  $T^3$ . We are going to state here without proof that  $H_1(T^3) \approx \mathbb{Z}^3$  and is generated by the three curves shown in figure 3.47. Now, for each of these three curves, we can create a barrier which counts the number of times a curve loops “in the same way” as this one (for an example of a closed curve looping multiple times “in the same way” as the blue curve, see figure 3.48). Before we were counting how many times a curve wound around a specific hole, but it is not immediately clear what a hole might be in this case (even though

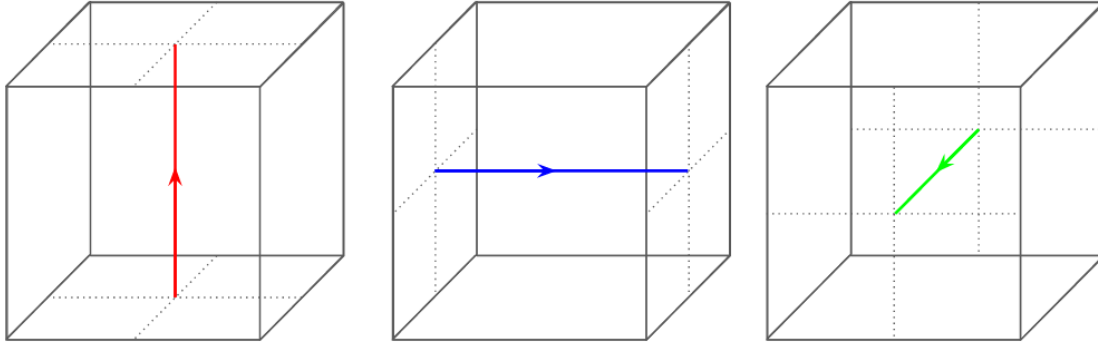


Figure 3.47: Three curves which generate the 1st homology group of the 3-torus

our interpretation of homology as hole-counting tells us there should be 3 2-dimensional holes).

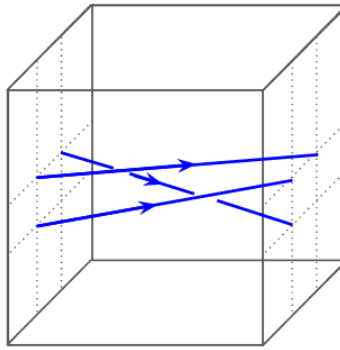


Figure 3.48: A curve which wraps around one of the “holes” three times

However, to create a barrier which counts the number of times a path crosses it, we must create a 2-dimensional barrier. In fact within an  $n$ -dimensional object, if we wanted to create a barrier to count  $k$ -dimensional intersections, we must create an  $n - k$ -dimensional barrier, which is where the  $n - k$  in the duality theorem comes in. So, we can see that these barriers will be of the form found in figure 3.49. Now we have an idea of what some elements of  $H^1(T^3)$  look like. Note however, that the barriers representing these cohomology elements are in themselves elements of  $H_2(T^3)$ . They have no boundary because what boundary they would have had got glued up around the edges, but they themselves are not boundaries of three-dimensional objects.

Similarly, (although this may be harder to visualize) we can see that if we wanted to define an element of the 2nd cohomology group (a cochain which assigns a number to surfaces) in this space, we could do it with an element of the first homology group. If an oriented surface (remember we can define an orientation using a clock in our surface) intersects with one of our lines, then since the surface and the line are oriented, we can determine if the embedded clock and the line obey the right-hand rule. If they do, assign that intersection a positive value, otherwise assign it a negative value. Then we can

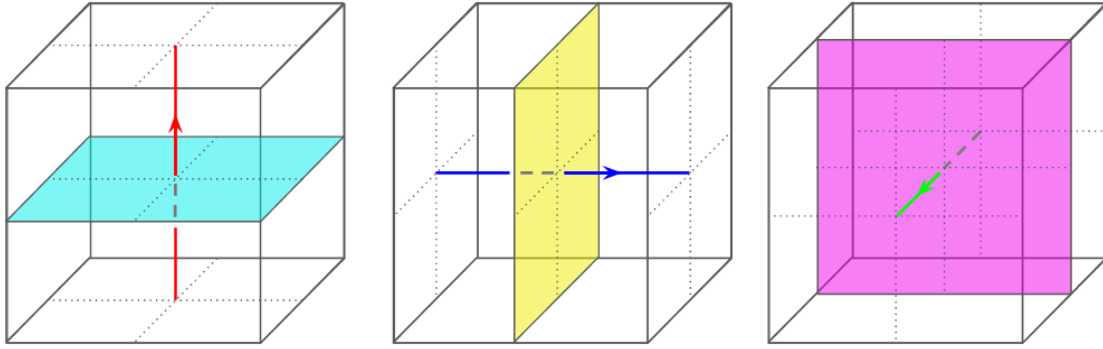


Figure 3.49: Homology and cohomology elements in  $T^3$

again see that elements of  $H^2(X)$  correspond to elements of  $H_1(X)$ .

### 3.8 Final Thoughts on Cohomology

Much of the idea, definitions, and method of calculation for cohomology is taken straight from homology. Indeed, most, if not all of the additional topics listed for homology have almost identical corresponding topics in cohomology (perhaps with some reversed arrows though). We have even seen that the cohomology groups of a space are always completely determined by the homology groups of that space. So, it raises the question, should we be studying cohomology if there is a similar, more foundational, and more accessible topic in homology? I hope that because of the last two sections, your answer is a resounding yes. As we saw with the universal coefficient theorem and Poincaré duality, the power in cohomology often is not inherent within cohomology, homology and cohomology become more powerful and more effortless when used in tandem. However, it should be mentioned that cohomology does have great power not on its own and there are many heavy theorems which rely on and are defined in terms of cohomology.

Unfortunately, we must soon leave the world of (co)homology, but before we do, the author suggests that if the reader is hungry for more, they should check out the section in [3] on the cup product. It gets slightly more advanced but offers a way in which cohomology on its own can provide some extremely interesting results.



# Using Topology to Study Algebra

We spent a lot of time talking about how to use algebra to study topological objects, but now it's time to switch gears a little bit. There are many ways to use topology to study algebra, but we will focus on one. First, we will find a space which somehow canonically represents the algebraic structure you are looking to examine. Then we will study the topological properties of that object. Since the space we chose is entirely dependent on the algebraic structure and unique (up to certain equivalences), we can define new invariants of algebraic structures based on the topological properties of the space in order to help us to distinguish it from another structure. However, the real magic happens when topological properties of the canonical space inform you about certain properties of your algebraic structure. In this section, the specific algebraic structures that we will be studying here will be groups and the canonical space we will be using is called a  $K(G, 1)$ . Then we will talk about two special properties obtained from the topological space, the geometric dimension and cohomological dimension and find out what they can tell us about the group.

## 1 $K(G, 1)$ Spaces: Putting the Group $G$ in Topology

Our goal in this section is to come up with a topological space which is canonically linked with a given group  $G$ . A natural way to progress with this is to find a space where either its fundamental group or one of its homology or cohomology groups is isomorphic to  $G$ . However, the homology groups and cohomology groups of a space are always abelian, so if  $G$  is not abelian, these groups will not work. Therefore, we will start by finding a space  $X$  whose fundamental group is isomorphic to  $G$ . However, there can be many such spaces. Even if  $G$  is just the trivial group, it turns out that for any  $n$  greater than 1,  $\pi_1(S^n)$  is trivial, so even for the simplest of groups, there are infinitely many distinct spaces whose fundamental group is isomorphic to  $G$ . We have to narrow down our space by choosing some additional criterion that the space must match.

## 1.1 Asphericity

The condition that we will choose is called *asphericity*. A space  $X$  is aspherical if it is path-connected and there is no way to embed a sphere of dimension 2 or greater in  $X$  which cannot be shrunk down to a point. Notice that we do not include the 1-dimensional sphere because that would mean that the fundamental group is trivial.

Let us look at some examples of aspherical spaces. We can see that if the space is  $\mathbb{R}^n$  then *any* sphere that we embed (not just the higher-dimensional ones) can be shrunk down to the origin by linearly interpolating between the origin and the points in the sphere. Another family of spaces which we have seen before which are aspherical are the  $n$ -dimensional tori,  $T^n$  for  $n > 1$ . These are created by taking an  $n$ -dimensional cube and gluing opposite faces together. It is fairly difficult to give a rigorous proof of why the tori are aspherical, but it has to do with the fascinating subject of *covering spaces*, which in a sense is a way of unraveling your space (it turns out that  $T^n$  unravels into  $\mathbb{R}^n$  which is why it is aspherical). However, you can take it on faith that  $n$ -tori are aspherical.

In addition, all surfaces besides the 2-sphere and real projective plane are aspherical. The sphere is not aspherical because if you embed a 2-sphere in the 2-sphere using the identity map, there is no way to bend or stretch the embedded sphere so that it shrinks down to a point (similar to if you surrounded an orange by a thin rubber membrane you could not get the membrane off without breaking it). Seeing the embedded sphere in the real projective plane is a little more difficult. We mentioned that one way to construct the real projective plane is to glue opposite points on a sphere. This is a valid way to map the sphere into the projective plane, but the fact that this mapping cannot be shrunk down to a point is much less trivial. However, this is plausible if we think about the fact that one of the great circles on this sphere is homotopic to the curve which cannot be shrunk down to a point (recall from the fundamental group that there is only one!). So if you could shrink the sphere down to a point then you could shrink that path down to a point.

## 1.2 Definition of a $K(G, 1)$ Space

A space which is aspherical, path-connected and has fundamental group  $G$  is called a  $K(G, 1)$  space (if it had fundamental group  $H$  it would be a  $K(H, 1)$ ). The 1 in the notation comes from the fact that all the homotopy groups other than the first homotopy group (the fundamental group) are trivial. If you had a space where every homotopy group was trivial except for the 4th group which was isomorphic to  $\mathbb{Z}$  it would be called a  $K(\mathbb{Z}, 4)$ . Now, these spaces are not entirely unique. Both the annulus and the circle are  $K(\mathbb{Z}, 1)$  spaces, but these two spaces are homotopy equivalent, meaning you can stretch,

bend, compress, or expand these spaces so that they coincide (this is not the technical definition which is much more complex and does not add much to intuition, so we will exclude it). Any two  $K(G, 1)$ 's are homotopy equivalent, so up to a certain point, they are the same (in particular, they have the same homotopy groups, homology groups, and cohomology groups).

Back in the chapter on fundamental group, we discussed (very briefly) a way to construct a space with a given fundamental group using the pancake theorem. We can use a method very similar to this to create a  $K(G, 1)$  space using something that I will refer to as the Russian tea cake theorem (it is a generalization of the pancake theorem to higher dimensions, so a higher-dimensional round cake seemed fitting). The Russian tea cake theorem says that gluing in an  $n$ -ball will make the boundary of that ball homotopy equivalent to a point. Note that the pancake theorem is a special case of the Russian tea cake theorem.

### 1.3 Construction of a $K(G, 1)$ Space

We can create a  $K(G, 1)$  for a specific  $G$  by constructing a corresponding CW-complex. Let's start by finding a presentation of  $G$ . We then take the 1-skeleton to be a bouquet of  $n$  circles (where  $n$  is the number of generators of the free group in the presentation) and then use the pancake theorem to add every relation in the presentation to create the 2-skeleton. Adding this may have created some non-trivial 2-sphere embeddings in your space. So, for each of those non-trivial 2-sphere embeddings, let us use the Russian tea cake theorem to glue in a 3-ball and create our 3-skeleton. Note that we do not add any elements to the fundamental group because for every loop which goes through one of these new 3-balls, we can just push it to the boundary. However, we might be worried about adding a relation to the fundamental group, and while we will not prove that fact here, one way to argue it is to again use the fact that any loop going through the sphere can be made to go to the surface. So for any curve which can be shrunk down to a point after gluing, we can do that same shrinking on the surface of the 3-ball.

After we have glued in these 3-balls however, we might have created some non-trivial 3-spheres in your space. So, we glue in some 4-balls to create the 4-skeleton! And so on and so forth as long as you have non-trivial spheres in your space. Sometimes you *always* have to keep on going, and in this case you just don't stop. You keep on adding balls in forever and eventually your space becomes infinite dimensional.

Let us begin the construction of this for  $\mathbb{Z}/2\mathbb{Z}$ . We know that the very beginning of the construction here will give us a real projective plane as the 2-skeleton. Then, using the visualization of the real

projective plane as a sphere with antipodal points identified, we can glue a 3-ball into the center of this sphere. This gives a solid 3-ball whose antipodal boundary points are identified. However, it is difficult to determine whether or not there is a non-trivial way to embed a 3-sphere into this space. We will eventually come to an answer to this, but for now, we will leave it for the reader to think on.

## 1.4 Moving Forward

Now that we have our canonical space for a given group, we can start studying properties of this group. For example, we can try to determine the homology and cohomology groups of this space (determining the fundamental group of this space should not be difficult). If we look at the homology groups of this space, we have entered the field called *group homology*. We can also look at whether or not this CW complex has finitely many balls in each of its  $n$ -skeletons, or whether or not it has infinitely many  $n$ -skeletons each of which has finitely many balls or if the total space is constructed using finitely many balls total. If we consider these properties, we are considering the *finiteness properties* of the group. If the reader is interested in either of these topics, the author suggests that they look at [2]. However, in the next chapter, we are going to look at a slightly different topic, dimensions of groups.

## 2 Dimensions of Groups: A Broad, but Not Deep Introduction

Once we have a  $K(G, 1)$  space for our group, we can talk a bit about the dimensions of that  $K(G, 1)$  space. We will talk about two different types of dimension, the *geometric dimension* of a group  $G$  (denoted by  $\text{gd}(G)$ ) and the *cohomological dimension* of  $G$  (denoted by  $\text{cd}(G)$ ). Most of this section is considered in much fuller generality and very delicately in [1].

### 2.1 Geometric Dimension

#### Dimension of a CW-complex

Before we define the geometric dimension of a group, we first need to define the dimension of a CW-complex. A CW-complex  $X$  is  $n$ -dimensional if it has an  $n$ -skeleton but no  $(n + 1)$ -skeleton. Now, we have looked 0, 1, 2, and 3-dimensional CW-complexes throughout this paper. The 0-dimensional object we looked at is the space consisting of 2-distinct points. We can see that it has a 0-skeleton (namely those points) but no 1-skeleton since there are no lines in the space. We looked at a couple of 1-dimensional spaces, the circle, and the bouquet of  $n$  circles. We have looked at quite a lot of 2-dimensional spaces, the annulus, the torus, the real projective plane, and the sphere just to name a

few. And finally, we looked at one 3-dimensional space, the 3-torus, even though we did not specifically construct it as a CW-complex.

### Definition of Geometric Dimension

The geometric dimension of a group  $G$  is the smallest number  $n$  such that there is an  $n$ -dimensional  $K(G, 1)$ . Note that since there are multiple different  $K(G, 1)$  spaces for a given group, we have to define it as a minimum. To see an example of this, let us look at the annulus and the circle. We mentioned that these two spaces are homotopy equivalent, and we know that the circle is aspherical (which actually implies the annulus is aspherical because asphericity is invariant under homotopy equivalence). So, these two spaces are both  $K(\mathbb{Z}, 1)$  spaces. However, we can see that the dimension of the circle is clearly 1 while the dimension of the annulus is 2. Therefore, since we cannot find a lower dimensional example of a  $K(G, 1)$  (the fundamental group of a 0-dimensional space is always trivial),  $\text{gd}(\mathbb{Z}) = 1$ .

### Geometric dimension 1

If  $G$  is a free group on  $n$  generators, then it has geometric dimension 1, since the bouquet of  $n$  circles is aspherical and has dimension 1. In fact the converse holds as well, if a group  $G$  has geometric dimension 1, then it is a free group. This is our first real result in topological group theory, and while this may seem obvious, the converse of this statement is quite difficult to prove and took many years to prove in full generality after it was conjectured. However, we will mainly be using it to show that certain groups have geometric dimension 2 (using this, we just need to show the group  $G$  is not free and that there is a 2-dimensional  $K(G, 1)$  in order to show that it has geometric dimension 2).

### Some properties of Geometric Dimension

The first major property is that if the  $n$ -th homology group of a  $K(G, 1)$  is not trivial then the group has dimension at least  $n$ . This makes sense since we have seen that if a space contains no  $n$ -chains its homology group is trivial. So, taking the contrapositive, if a space has a non-trivial  $n$ th homology group then it contains an  $n$ -chain and so has dimension at least  $n$ . But since all  $K(G, 1)$  spaces have the same homology groups, this implies that *every*  $K(G, 1)$  has dimension at least  $n$  and so  $\text{gd}(G) \geq n$ .

The next property which we will be talking about is a bit surprising. Every non-trivial finite group has infinite geometric dimension. This means that for a finite group which is not the trivial group, you cannot find a  $K(G, 1)$  which is finite dimensional. When you are constructing it, it you have to continue gluing balls in forever. The proof of this fact is quite non-trivial, but [2] is a good starting point.

Our third and final property of geometric dimension is that it does not increase when you pass to a subgroup. Basically if  $H \leq G$  then  $\text{gd}(H) \leq \text{gd}(G)$ . In particular, this means that every group with a finite subgroup has infinite geometric dimension.

### Some Examples of Geometric Dimension

The first group we will examine is  $\mathbb{Z}^n$ . The reader should be warned that because of the difficulty in calculating geometric dimension and fundamental group, most of this section will have to be taken on faith. The first big assumption that we will be making is that  $T^n$  is a  $K(\mathbb{Z}^n, 1)$ . A proof of this using the pancake theorem would be nasty (you would have to start with a bouquet of  $n$  circles and then glue in  $aba^{-1}b^{-1}$  for every  $a$  and  $b$  in your  $n$  circles and then glue in more balls until you get up to dimension  $n$ ). The most natural way to do this is by using the Cartesian product  $\times$  of topological spaces, which we have not covered. However, we can see from this that the geometric dimension of  $\mathbb{Z}^n$  is at most  $n$  since we have found an  $n$ -dimensional  $K(\mathbb{Z}^n, 1)$ . Then we can use Poincaré duality to show that  $T^n$  has a non-trivial  $n$ th homology group. Since  $T^n$  is an  $n$ -manifold,  $H_n(T^n) \approx H^0(T^n)$  and since  $T^n$  is path-connected,  $H^0(T^n) \approx \mathbb{Z}$ . Therefore,  $H_n(T^n) \approx \mathbb{Z}$  and so  $\text{gd}(\mathbb{Z}^n) \geq n$ . So, since  $\text{gd}(\mathbb{Z}^n) \geq n$  and  $\text{gd}(\mathbb{Z}^n) \leq n$ ,  $\text{gd}(\mathbb{Z}^n) = n$ .

The next example we will mention is  $\mathbb{Z}/2\mathbb{Z}$ . We started trying to create a  $K(\mathbb{Z}/2\mathbb{Z}, 1)$ , but we gave up pretty quickly when we did not know what a 3-sphere looks like. It is pretty fortuitous that we stopped there because we could have been there quite a long time, forever in fact. We just mentioned that every finite group is infinite dimensional so  $\mathbb{Z}/2\mathbb{Z}$  is infinite dimensional and so any  $K(\mathbb{Z}/2\mathbb{Z}, 1)$  must be infinite dimensional. In particular, the one we were trying to construct is infinite dimensional. Let us give a bit of an idea why that might be. First, let us define  *$n$ -dimensional real projective space*.  $n$ -dimensional real projective space is defined by taking the  $n$ -dimensional ball and identifying antipodal boundary points (this is the definition we will be using although there are many, many definitions). We will also define it equivalently by taking an  $n$ -sphere and identifying antipodal points. So, we can see that  $\mathbb{R}P^1$  is a circle, because you take a 1-ball (a line) and identify antipodal boundary points (the line endpoints). We can also take a 1-sphere (a circle) and glue opposite points. To do this, think about double-wrapping a rubber-band (the way you might if you are trying to wrap a deck of cards but the rubber band is too loose). After you are done, you are yet again left with a circle.  $\mathbb{R}P^2$  is a disk with antipodal boundary points glued together or a sphere with antipodal points glued together.  $\mathbb{R}P^3$  is gotten by taking a 3-sphere (which can be obtained by gluing the boundary of two 3-balls together) and gluing antipodal points (gluing the interiors of the three balls together as well) or by taking a 3-ball and gluing opposite boundary points to each other. And so on and so forth...

So, if we start with  $\mathbb{R}P^1$  looking at it as a 1-sphere with antipodal points glued and glue in a 2-ball, we get a 2-ball with opposite points glued together, that is,  $\mathbb{R}P^2$ . Then if we think of  $\mathbb{R}P^2$  as a sphere with antipodal points glued together, we can glue in a 3-ball to get a 3-ball with antipodal boundary points glued together, i.e.  $\mathbb{R}P^3$ . Then we have to think of it as a 3-sphere with antipodal points glued together and so on... Using these two interpretations, we can see that at every step we get a new non-trivial way to map a sphere, namely by thinking of  $\mathbb{R}P^n$  as a sphere with antipodal points identified. So to make a  $K(\mathbb{Z}/2\mathbb{Z}, 1)$  we really do have to go on forever. At the end, we get  $\mathbb{R}P^\infty$  where you take an  $\infty$ -sphere and glue antipodal points. However, the  $\infty$ -sphere is the set of all summable sequences where the sums of the squares of the entries is 1. This space is a mess, so we will leave it be.

There are two main steps to determine if the geometric dimension of a group  $G$  is  $n$ :

1. You must show  $\text{gd}(G) \geq n$ .
2. You must show  $\text{gd}(G) \leq n$ .

The first step is incredibly hard, but we have some tools at our disposal. Mainly, we can find a  $K(G, 1)$  (let us call it  $X$ ) for  $G$  and determine if  $H_n(X)$  or  $H^n(X)$  are non-trivial. We can also look at the subgroups of the spaces and see if they have geometric dimension  $n$ . To help with the second step, we have to construct a  $K(G, 1)$  which is  $n$ -dimensional, a task which is quite daunting especially when it can be quite difficult to construct a  $K(G, 1)$  for your space in the first place. We have a method to do so, but it involves finding all of the non-trivial spheres in your space at each dimension, a task which is almost equivalent to finding all of the homotopy groups of the space.

Luckily, in the next part of this chapter, we will be given a tool to help with this incredibly difficult second step.

## 2.2 Cohomological Dimension

The *cohomological dimension* of a group is defined a bit more simply than the geometric dimension of a group. The cohomological dimension of a group  $G$  (denoted  $\text{cd}(G)$ ) is the last point where  $H^n(K(G, 1))$  is non-trivial. Note that we can just refer to  $K(G, 1)$  since any two  $K(G, 1)$  spaces will have the same cohomology groups. Actually, there is a small note that should be made here. In cohomology, you could choose to alternatively define the  $n$ -cochains to assign  $n$ -chains other numbers than integers. To determine the cohomological dimension, you must find the last point where  $H^n(K(G, 1))$  is non-trivial using any number system. However, that is the last we will talk about that.

Even with looking at the other number systems, the cohomological dimension of a group is oftentimes much easier to calculate (in fact there is a way to do it without using a topological space at all!).

Furthermore, we can get a nice relationship between cohomological and geometric dimension right off the bat, mainly  $\text{cd}(G) \leq \text{gd}(G)$ . This is because if  $\text{cd}(G) = n$  then  $H^n(K(G, 1))$  is non-trivial, so there must be some non-trivial  $n$ -chain to assign a number to. From this fact, we can get that all  $K(G, 1)$  have a non-trivial  $n$ -chain and so are dimension at least  $n$ . Therefore,  $\text{gd}(G) \geq n \geq \text{cd}(G)$ . However, there is another incredible relationship (presented here without proof). If  $\text{cd}(G) \geq 3$  then  $\text{gd}(G) \leq \text{cd}(G)$ . From this, we get that if  $\text{cd}(G) \geq 3$  then  $\text{cd}(G) = \text{gd}(G)$ .

There are actually so many relationships between  $\text{cd}(G)$  and  $\text{gd}(G)$  that the only time they might not be equal is when  $\text{cd}(G) = 2$  and  $\text{gd}(G) = 3$ . And we don't even know if this is possible! This brings us to our first (and only) open problem discussed here, the Eilenberg-Ganea conjecture, a fitting way to end this paper as we now delve into current research topics:

**Conjecture** (Eilenberg-Ganea). *If  $\text{cd}(G) = 2$  then there is a 2-dimensional  $K(G, 1)$ , i.e.  $\text{gd}(G) = 2$ .*



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