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The Central Hankel Transform

Matthew J. Levine
Colby College

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Matthew Levine
Advisor: Ben Mathes

Introduction

If pigs have wings, then every number is the sum of two primes[1]

– Gouvea, History of Math (236)

The category of "Hankel," in history, operator theory, and matrix algebra, is rich and fascinating; this thesis, in congruence with a paper that has been submitted to publication[2], hope to add to that legacy of unexpected and useful mathematical phenomena. In history, Hankel was an ephemeral but powerful influence on 19th century German mathematics. His class of symmetric matrices and their underlying operators became a useful numerical and analytical tool that was studied over the next few centuries. We summarize the life of Hermann Hankel, describe the research already committed to matricial Hankel theory, enumerate our own original results, and finally turn back to outline the underlying operator theory in terms of surrounding functional analysis.

[1] This quotation is purposefully taken out of context for comical effect. Gouvea gave it as an example of the logical conjunction fallacy

[2] In a paper written by primary author Professor Ben Mathes, also by Professor Justin Sukiennik, and Liam Connell Colby class of 2015

0. History

[Weierstrass] did not care about his studies, and he spent four years of intensive fencing and drinking

– O'Connell

The history of the Hankel Transform is rooted in that of its namesake, Hermann Hankel. In this section, we describe Herman Hankel and his diverse, prolific contributions to the general field of mathematics. Hankel's success is observed as the direct sum of the spaces of education represented by Weierstrass, Kronecker, and Möbius in the same way that the Möbius, Kronecker, and Weierstrass theorems will coalesce into a powerful theory of Hankel operators in later sections.

Herman was borne in 1839 to a physicist in southern Germany (Ditzian). Exposed to the intellectual and academic elite at a young age, he thrived, studying with his father, and moving from one prominent mathematician at the time to another. He studied with Möbius, then Riemann, and in 1861 found himself at Berlin studying with Kronecker and Weierstrass (O'Connor), two of the most preeminent mathematicians of the time. In order to adequately understand the roll of Herman Hankel, we have to understand the influences of the great minds he surrounded himself with.

Table 1 Photos of relevant mathematicians in the timeline of Hermann Hankel (O'Connor)

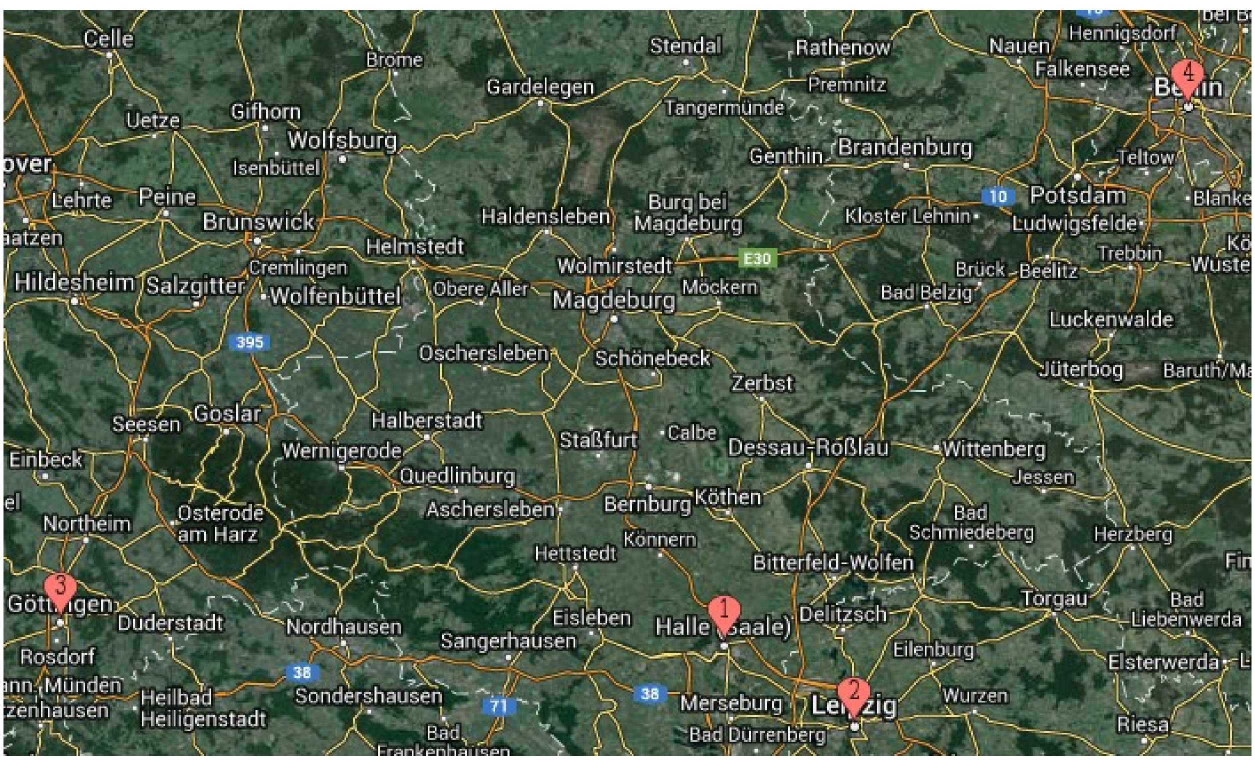
Hankel	Möbius	Riemann	Kronecker	Weierstrass
				

While Hankel passed away at the young age of 34 (Chang), his first (non-parental) professor, August Möbius, was inventive deep into his halcyon days. In fact, Möbius was 67 years old when Hankel arrived

at his doorstep in Leipzig, a year after the professor had discovered Möbius Strip*,* in 1858 (O'Connor). Möbius, like Hankel, had diverse interests – he developed barycentric calculus (think barycentric coordinates, for use in, for example, filling polygons in computer graphics) (Pickover). He is credited with numerous puzzling equations still in use today, like the Möbius Function that has found its way in subatomic particle theory but remains poorly understood (Pickover).

By 1860, Hankel was ready to move on, and found himself in the presence of the great Bernhard Riemann at the height of Riemann's mathematical career (O'Connor). Riemann had been promoted to the head of mathematics at Gottingen just one year prior (Chang) and Riemann geometry was only six years old (Gouvea); Hankel's timing, like always, was impeccable. Although Hankel was only with Riemann for one year, Riemann's influence of Hankel's work is dramatic.

Figure 1 Map of Herman Hankel's academic journey. Point 1: Hankel is born (1834). Point 2: Hankel and Möbius join forces (1857). Point 3: Riemann joins Hankel's resume (1860). Point 4: Hankel embraces the University of Berlin (1861)



In 1860, Hankel fulfilled the German tradition of completing a doctoral degree at a different University than the one at which he began. He joined the University of Berlin's academic force under the titanic

leadership of Weirstrass and Kronecker (O'Connor) only a year after Weirstrass had published his proof of the intermediate value theorem (Pickover). Weirstrass' lectures were famous for imparting in his pupils the rigor and exactness of mathematics that made Weirstrass, himself, famous (Gouvea) – had they been born in the same era, Professor Weirstrass and Professor Livshits might have enjoyed assigning the same problem sets to their students. Weirstrass' work is some of the most foundational in Hankel's initial work, of which we are most interested; the Weirstress compactness theorem is what allows us to start talking about Hankel operators on a Hilbert space in the first place (Partington).

Hankel's doctoral thesis at Berlin is the exact paper that interests us – "A special case of symmetric determinants", published in 1861 only year after arriving at the university[3] (Chang). He continued at a prolific rate, publishing almost a paper a year, on average, even posthumously for some time. The influence of his tutors is evident on the operator theory side, with the "Kronecker" and "Möbius" theorems being integral to the implementation of the isomorphism between Hardy and Hilbert spaces (Partington).

Hankel's interests were, like his educators, eclectic. He contributed substantially to the field of complex numbers, including papers published in 1867 onward and a series of lectures. He also considered the fields of complex analysis, math history, functional analysis and operator theory to be his own. An identifying feature of his historic works, for example, is the credit he gives to Arabic nations in preserving and forwarding mathematics through the middle ages. (Dauben)

Figure 2 Tabled timeline of Hankel's publications. All papers published after 1873 were posthumously published

1861	On the general theory of the motion of liquids
1861	**A special class of symmetric determinants**
1863	The Euler integral with infinite argument variability
1864	Mathematical determination of optics (?)
1867	Lectures on complex functions
1867	Theory of complex number systems: in particular, imaginary numbers and Hamiltonian quaternions, along with their geometric representations
1867	Lectures on complex numbers and their functions (part 2)

[3] (Iohvidov) notes that the theory of Hankel matrices as we consider it probably starts gaining traction from a series of papers from 1884-1912

1867	Theory of complex number systems
1869	First and second order cylindrical functions
1874	To the history of mathematics in antiquity and the middle ages
1875	The Fourier series and integrals over cylindrical functions
1875	The elements of projective geometry in synthetic treatment
1875	Estimating Integrals of cylindrical functions
1882	Investigations of infinitely oscillating and discontinuous functions
1885	The development of mathematics in the last centuries

Although he contributed to many fields, Hermann has critiques in almost all of them – though no one seems to agree about exactly what to criticism him. Cantor was highly critical of his posthumously published work on math history (Dauben). With a grain of cynicism, however, we might accuse Cantor (one of the most famous math historians of the time) as having a conflict of interest here – Hankel had been a direct student of Kronecker, and "to Kronecker and those who shared his view, Cantor's work was a dangerous mixture of heresy and alchemy" (Gouvea). On the other hand, later authors were considerate of his math history and functional analytical work, but thought his complex analysis was second rate (O'Connor) . He may never have pleased everyone, but Hermann Hankel lived a powerful and diverse, if concise, life. His legacy, as evidenced by the existence of this document, still persists today.[4]

[4] I regret that I was not able to read much of Hankel's original work, as it is in German and minimally translated. I have translated using electronic tools what I can, and otherwise relied on secondary sources for this section.

1. Hankel Matrices

In this section, we give a brief overview of Hankel Matrices, known results pertaining thereof, and a summary of their applications. The section at the end of the paper, "Hankel Operators," gives more exposition on the theoretical side of Hankel Matrices.

I cannot possibly cover the 200 years of research done in this area in a few pages; I will strive to present the most important and interesting theorems measured with respect to the objectives of the independent research conducted.

In particular, we cover the definition of the Hankel transform and the consequences of taking the transform of Catalan and Fibonacci numbers. We describe some of the more advanced tools in Hankel analysis, like canonical forms for the matrix and rank-analysis tools.

Definition 1.0.1 Determinant (Brescher) (Iohvidov) (Axler)**,** e.g.[5]

The *determinant* of a square matrix $A \in \mathbb{M}_n(\mathbb{C})$ is given by the Leibniz formula:

$$\det(A) = \sum_{\sigma} sgn(\sigma) \prod_{i=1}^{n} A_{i,\sigma_i}$$

Where the sum is computed over all σ permutations of the set $\{1, \dots, n\}$. This formula produces the same scalar as Laplace (or "Cofactor") expansion described in any of the cited sources. The following are facts about determinant

a) $\det(A) = \det(A^T)$
b) The $\det(A) = 0$ if and only if
   - A is called *singular* (otherwise it is nonsingular)
   - A is not invertible
   - The columns of A are not linearly independent
c) Column swaps do not affect determinant
d) Adding a scalar multiple of one column into another column does not affect determinant
e) Scaling a column by α scales the determinant by α
f) Determinant is multilinear
g) Determinant is multiplicative
h) $\det(I_n + A_1A_2) = \det(I_n + A_2A_1)$ (Sylvester's Lemma)
i) The determinant of a triangular matrix is the product of its diagonals

[5] Where (Brescher) and (Axler) give thorough introduction to determinants in context, (Iohvidov) enumerates a large list of determinant-rules for practical use. For a simple presentation of these statements without proof, see (Hogben)

By 1.0.1a, every statement about columns is true about rows.

Definition 1.0.1 Hankel Matrix (Horn)

A matrix $A \in \mathbb{M}_{n+1}$ of the form

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_n \\ a_1 & a_2 & a_3 & \cdots & \cdots & a_{n+1} \\ a_2 & a_3 & \ddots & & & a_{n+2} \\ & & & \ddots & & \\ \vdots & . & & & \ddots & \vdots \\ & a_n & & & & \\ a_n & a_{n+1} & a_{n+2} & \cdots & \cdots & a_{2n} \end{bmatrix}$$

Is called a *Hankel Matrix*. The general term $A_{i,j} = a_{i+j-2}$ for some given sequence $a_0, a_1, \ldots a_{2n-1}, a_{2n}$. We will use the notation $H(\{a_i])$ tp denote the Hankel matrix of such a sequence a.

Definition 1.0.2 Hankel Transform

The *Hankel Transform* of a sequence $s = \{s_0, s_1, \ldots\}$ is the sequence of determinants of n-dimensional Hankel matrices generated from the fisrt n terms of s.

The Hankel transform, is not, in general, injective (see (Ehrenborg), for example)

■

Over the course of this section we will look for interesting – often unexpected – consequences of looking at the Hankel transform of various sequences. Among the most famous integer sequences is the Fibonacci numbers; one might be immediately motivated to suspect interesting behavior of the Hankel transform of the Fibonacci numbers. Although that person would be wrong, we will present alternative ways of generating Fibonacci numbers through Hankel transforms.

Example 1.0.1 Fibonacci Numbers in Hankel

Let the Fibonacci numbers be the sequence $f = \{0,1, f_2, f_3, \ldots\}$ where for $n > 1$, $f_n = f_{n-1} + f_{n-2}$.

$$H(f) = \begin{bmatrix} f_0 & f_1 & f_2 & \cdots \\ f_1 & f_2 & f_3 & \cdots \\ f_2 & f_3 & f_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Since adding columns does not affect determinant, we can consider the determinant of the matrix H' generated from $H(f)$ after subtracting the first two columns from the third. We would have

$$H_3' = \begin{Bmatrix} f_2 - (f_0 + f_1) \\ f_3 - (f_1 + f_2) \\ f_4 - (f_2 + f_3) \\ \vdots \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{Bmatrix}$$

Since the third column is not linearly independent from the first two, the $\det(H(f)) = \det(H'(f)) = 0$. In fact, we note that we could use any three consecutive columns to reach that conclusion. The corollary is that $rank(H(f)) = 2$.

■

It is interesting to point out that a reader particularly familiar with Fibonacci numbers would probably have already encountered the Hankel transform of the Fibonacci numbers, in a restricted form. The key is that $\det(H_n(f)) = 0$ only when $n \geq 3$; we can infer a lot just from the minimal case when $n = 2$.

Lemma 1.0.1 Fibonacci Generator (Grimaldi)

If $Q = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix}$, then $Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$ for $n \geq 1$. This is a trivial exercise in induction. It is worth noting that both Q and Q^n are Hankel.

Lemma 1.0.2 Cassini's Identity (Grimaldi)

Consider the Hankel transform of the "reverse" Fibonacci numbers, f^{-1} – the finite n-term sequences $\{f_{n-1}, f_{n-2}, \dots f_0\}$. In particular, consider $H_2(f^{-1})$ given by

$$H_2(f^{-1}) = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix}$$

So

$$\det(H_2(f^{-1})) = \det(\begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix}$$
$$= f_2 f_0 - f_1 f_1 = -1$$

In general,

$$\det\left(\begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}\right) = f_{n+1} f_{n-1} - f_n^2$$

But by Lemma 1.0.1 (accepting the definition of Q used in that lemma),

$$\det(H_n(f^{-1})) = \det(Q^n)$$
$$= \det(Q)^n$$
$$= (-1)^n$$

So

$$f_{n-1} f_n - f_n^2 = (-1)^n$$

That relation is known as Cassini's identity. We could also look at $\det(Q)$ by noting that

$$\det\left(\begin{bmatrix} f_{m+n+1} & f_{m+n} \\ f_{m+n} & f_{m+n-1} \end{bmatrix}\right) = \det(Q^{m+n})$$
$$= \det(Q^m Q^n)$$
$$= \det(Q^m)\det(Q^n)$$
$$= \det\left(\begin{bmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{bmatrix}\right)\det\left(\begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}\right)$$

In expanded form,

$$f_{m+n+1}f_{m+n-1} - f_n^2 = (f_{m+1}f_{m-1} - f_m^2)(f_{n+1}f_{n-1} - f_n^2)$$

■

Lemma 1.0.3 Eigenvalues of $H_2(f)$ (Grimaldi)

The eigenvalues of $Q_2(f)$ are ϕ and ψ, the golden ratio and reciprocal golden ration famous for being associated with Fibonacci identities. This follows trivially by definitional expansion of eigenvalues[6].

■

Definition 1.0.4 Binomial Transform

If $s = \{s_0, s_1, \ldots s_n\}$ is a sequence, then the *binomial transform* B of s is the set $s' = \{s_n'\}$ where

$$s_n' = \sum_{k=0}^{n} (-1)^k \binom{n}{k} s_k$$

Where $\binom{n}{k}$ denotes the binomial coefficient. The *invert transform* of s is the set $s' = \{s_n'\}$ where

$$s_n' = \frac{1}{1 + \sum_{i=0}^{n} s_-}$$

Definition 1.0.5 Invert Transform (Abrate)[7]

The *invert transform* I of a sequence $s = \{s_n\}$ is a sequence s' given by

$$\sum_0^\infty s_n' t^n = \frac{\sum_{n=0}^\infty s_n t^n}{1 - t\sum_{n=0}^\infty s_n t^n}$$

Lemma 1.0.5 Hankel Invariances (Layman)

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[6] We haven't defined eigenvalues so far, but choose not to since they only appear in this one note. See (Brescher), for example, for full treatment, or any of the sources listed on the definition of determinant

[7] I am not satisfied with this citation. While it provides a good definition of the Invert transform, it is not an article relating directly to this paper, where all other cited sources were chosen for their contribution to this thesis. The OEIS crowd - (Layman), (Cvetkovic), (Bouras), etc., provide a dead link a general reference to the OEIS website, which is unusable. The link http://www.research.att.com/~njas/sequences/ provided in (Layman) and https://oeis.org/wiki/Invert_transform linked on the OEIS website are dead as of 04/27/2015. The citation I've provided, while accurate, is not even temporarily useful, coming after other citations in this paper that rely on it.

The Hankel transform of a sequence is invariant under B and I. In other words, $H_n(s) = H_n\big(B(s)\big) = H_n\big(I(s)\big)$.

Definition 1.0.3 Catalan Numbers (A000108)

The Catalan numbers are the sequence $C = \{c_n\}$ where

$$c_n = \frac{(2n)!}{(n+1)!\,n!}$$

This definition is ubiquitous, and is equivalent to hundreds of others that come up in combinatorics problems, including such ostensibly disjoint topics as inserting n pairs of parentheses to a word of $n+1$ letters and counting the number of labeled dissections of a disk.

Lemma 1.0.4 Catalan Numbers (Layman)

Where C denotes the sequence of Catalan numbers, $H_n(C) = 1$ for all n.

Corollary 1.0.1 Catalan and Binomial (Cvetkovic)

The determinant of the binomial transform of the Catalan numbers is 1 for all n

Lemma 1.0.5 Catalan and Hankel (Cvetkovic)

If $C_k = \{c_k, c_{k+1}, \dots\}$, then $\det\big(H_n(C_k)\big) = 1$ for all n, k. Furthermore, if $S_k = \{s_k, s_{k+1}, \dots\}$ and $\det\big(H_n(S_k)\big) = 1$ at least for $k = 0, k = 1$ and for all n, then $s_n = c_n$ must be the Catalan numbers. In other words, <u>*the Hankel transform of the Catalan numbers characterizes that sequence.*</u>

Lemma 1.0.6 Generalized Layman's Conjecture (Bouras)

The determinant $H_n(\{c_{i+1} + c_{i+2}\})$ generates the even Fibonacci numbers as n varies. The determinant $H(\{c_i + c_{i+1}\})$ generates the odd Fibonacci numbers as n varies. These two properties characterize the Catalan numbers.

■

There were effectively two directions to take (Cvetkovic)'s work. In one direction, (Bouras) showed that the statement (Cvetkovic) made was stronger than he had made it. On the other, (Dougherty) extended the statement to include up to four terms of Catalan numbers. It is, as far as I know, still an open question what happens past four-sums of Catalan numbers.

Example 1.0.3 Hankel Canonical Form (Iohvidov)

If A is of Hankel quadratic form

$$A(x,x) = \sum_{j+k=0}^{n-1} s_{j+k}\,\xi_k \xi_k$$

i.e., H is generated by a sequence $s = \{s_i\}$ to be

$$H = \begin{bmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ \vdots & \vdots & \dots & \vdots \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{bmatrix} = \begin{bmatrix} a & a+d & \dots & a+(n-1)d \\ a+d & a+2d & \dots & a+nd \\ \vdots & \vdots & \dots & \vdots \\ a+(n-1)d & a+nd & \dots & a+(2n-2)d \end{bmatrix}$$

Then A can be canonically written as

$$A(x,x) = \frac{\left[\sum_{j=0}^{n-1}(a+jd)\xi_j\right]^2}{a} - \frac{\left[a\sum_{j=0}^{n-1}(a+(j+1)d)\xi_j - (a+d)\sum_{j=0}^{n-1}(a+jd)\xi_j\right]^2}{ad^2}$$

For all $a \neq 0, d \neq 0$.

Example 1.0.4 Extensions of Hankel Matrices (Iohvidov)

An *extension* of a Hankel matrix is constructed by embedding the matrix in the top left corner of a larger matrix. When this construction has the same rank as the embedded matrix, it is called a *singular extension*. Nonsingular Hankel matrices have infinitely many singular extensions of order $n+1$. A singular matrix with a nonsingular principle minor[8] has unique singular extension which can be formulaically determined.[9]

Lemma 1.0.2 Kronecker's Theorem (Iohvidov)[10]

If H_∞ is an infinite Hankel matrix of finite rank ρ, then its $\rho - 1_{th}$ leading minor is nonzero.

■

[8] Thinking in terms of cofactor expansion, a minor is the determinant of a cofactor. If a deletion row index and column index are the same, a minor is principle; if it's in the top-leftmost corner, it's leading. Principle leading combines the two. (Abadir)

[9] There are several interesting theorems about the "characteristic r-k numbers" of a Hankel matrix, particularly in determining its rank. Unfortunately, they are tangential to our work and elaborate to explain, but see (Iohvidov) if you're interested

[10] I have cited (Iohvidov), though the original proof evidently belongs to Kronecker from 1881. I can't read German, otherwise I would cite his paper directly

2. The Central Hankel Transform: Properties and Implications

In this, the main section of the paper, we define and investigate the Central Hankel Matrix , a sub-classification of the Hankel Matrix, leading to new identities for the Fibonacci Numbers. We show that the related Central Hankel Transform generated by the Fibonacci numbers with an offset uniquely defines the sequence. We demonstrate a closed formula for the Central Hankel Transform of other common sequences such as the Lucas numbers and simple arithmetic sequences.

2.1 Evaluating Common Sequences

Definition 2.1.1 Central Hankel Matrix

Let the Central Hankel Matrix of the integer sequence $A = \{a_0, a_1 a_2 \dots\}$be the infinite matrix $H = [h_{i,j}]$

$$H = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ a_4 & a_3 & a_2 & a_1 & a_0 & \dots \\ a_3 & a_2 & a_1 & a_0 & a_1 & \dots \\ a_2 & a_1 & a_0 & a_1 & a_2 & \dots \\ a_1 & a_0 & a_1 & a_2 & a_3 & \dots \\ a_0 & a_1 & a_2 & a_3 & a_4 & \dots \end{bmatrix}$$

such that $h_{i,-i} = a_0$ and $h_{i,i-i+k} = a_{|k|}\ \forall k \in \mathbb{N}$.

The persymmetric Central Hankel Matrix H_n of order n is the lower left $n\ x\ n$ submatrix of H.

Definition 2.1.2: Central Transform

The determinant of H_n is called the *Central Hankel Transform*, or *Central Transform*

■

Definition 2.1.3: Toeplitz Matrix

Let the *Toeplitz Matrix* of the integer sequences $A = \{a_0, a_1, \dots\}$ be the infinite matrix $T = [t_{i,j}]$

$$T = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ a_1 & a_0 & a_1 & a_2 & \dots \\ a_2 & a_1 & a_0 & a_1 & \dots \\ a_3 & a_2 & a_1 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Such that $t_{i,j} = a_{|i-j|}$ for all $0 \leq i \leq j$. The finite Toeplitz matrix T_n is the upper left submatrix of T; the *Toeplitz Transform* is the sequence of determinants of T_n as n varies. [11]

Lemma 2.1.3: Central Hankel Matrix and Toeplitz (Sukiennik)

$$det(H_n) = (-1)^{n(n-1)/2} \det T_n$$

The multiplication of a Toeplitz Matrix with an anti-diagonal matrix $\begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$ yields the central Hankel matrix. The determinant is then evaluated by multiplicity where the sign term is the determinant of an anti-diagonal matrix.

Theorem 2.1.1: Fibonacci and Central Transform

Let $F = \{0,1, \dots, f_{n-2} + f_{n-1}, \dots\}$ be the Fibonacci numbers and $H_n(F)$ be the Hankel matrix generated by F.

$$\det(H_n(F)) = 2^{n-2}(-1)^{(n-1)(n-2)/2}$$

We present two proofs of this theorem. The first is a more concise version of the second, but the second gives an interesting case in which we are actually able to evaluate the cofactors of a matrix explicitly. We will not bother tracking the sign in the second proof, since the proof is given for exposition and the sign can be handled, as shown in the first proof.

Proof 1:

Consider subtracting the $n - 1_{th}$ and $n - 2_{nd}$ columns of H_{n+1} into the n_{th} column of H_{n+1}. The n_{th} column of the resultant matrix would be the vector v given by

$$v = \{(f_0 - f_1 - f_2), (f_1 - f_0 - f_1), \dots, (f_i - f_{i-1} - f_{i-2}), \dots, (f_n - f_{n-1} - f_{n-2})\}$$
$$= \{-2,0, \dots ,0, \dots ,0\}$$

The resultant matrix generated through column operations, and thus has the same determinant as H_{n+1}. Similarly, we use v to eliminate the top row through column operations without affecting the determinant. Apply Laplace expansion we see that the one non-trivial minor is H_n, and by recursion we have that $H_n(F) = (-1)^{(n-1)(n-2)/2} 2^{n-2}$.

■

Proof 2:

Let A_i denote the i_{th} cofactor of the expansion of H_{n+1} generated by eliminating the top row, i.e.,

$$\det(H_{n+1}) = f_n A_n - f_{n-1} A_{n-1} + \dots + f_0 A_0$$

We will show that $\forall A_i : i \neq 0,1,\ or\ 2,\ |A_i| = 0$

Consider some cofactor $A_i : i \neq 0,1, or\ 2$. A_i must have the columns

[11] My literature review was of Hankel Matrices, but all major sources, e.g., (Iohvidov), also cover Toeplitz matrices; many conclusions are redundant. We discuss the Hankel Transform in depth instead of the Toeplitz transform because we look to parallel the results found about Hankel matrices.

$$col_{n-1} = \begin{bmatrix} f_1 \\ \dots \\ f_n \end{bmatrix}, \quad col_{n-2} = \begin{bmatrix} f_0 \\ \dots \\ f_{(n-1)} \end{bmatrix}, \quad col_{n-3} = \begin{bmatrix} f_1 \\ \dots \\ f_{n-2} \end{bmatrix}$$

Without affecting the determinant, we can make the assignment $col_{n-3} = (col_{n-3} + col_{n-2}) - col_{(n-1)}$ which reduces to the zero vector after applying the Fibonacci explicit values to the top element and its recursive definition to the rest. Thus, $\det(A_i) = 0 \ \forall i \notin \{0,1,2\}$

We now consider the three remaining cofactors, $A_0, A_1, and\ A_2$. As $f_0 = 0$, the trivial case follows that $f_0 A_0 = \emptyset$. Consider A_1 and A_2:

$$A_1 = \begin{bmatrix} f_{n-1} & f_{n-2} & \dots & f_1 & f_1 \\ f_{n-2} & f_{n-1} & \dots & f_0 & f_2 \\ \dots & \dots & \dots & \dots & \dots \\ f_0 & f_0 & \dots & \dots & f_n \\ f_0 & f_2 & \dots & f_{n-2} & f_n \end{bmatrix}$$

$$A_2 = \begin{bmatrix} f_{n-1} & f_{n-2} & \dots & f_0 & f_1 \\ f_{n-2} & f_{n-1} & \dots & f_1 & f_2 \\ \dots & \dots & \dots & \dots & \dots \\ f_0 & f_0 & \dots & \dots & f_n \\ f_0 & f_2 & \dots & f_{n-1} & f_n \end{bmatrix}$$

Consider adding $(-1) \cdot col_{n-1}$ into col_{n-2} of A_1, yielding

$$A_1 = \begin{bmatrix} f_{n-1} & f_{n-2} & \dots & f_0 & f_1 \\ f_{n-2} & f_{n-1} & \dots & f_1 & f_2 \\ \dots & \dots & \dots & \dots & \dots \\ f_0 & f_0 & \dots & \dots & f_n \\ f_0 & f_2 & \dots & f_{n-1} & f_n \end{bmatrix}$$

We note that $\det(A_1) = -\det(A_2)$. Negating the column, we have transformed A_1 into A_2 while preserving the determinant of both; the two matrices have the same determinant.

By subtracting the second to last column of A_1 from the last column of A_1 with have that $A_1 = H_n$ which directly implies that $\det(A_1) = \det(H_n)$. As $\det(H_{n+1}) = 2\det(A_1)$, we have that

$$|H_{n+1}| = 2^{n-2} * |H_2|$$

$|H_2|$ is given by $\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$, allowing us to conclude that

$$|H_n| = 2^{n-2}$$

■

Theorem 2.1.2: Lucas and Central Transform

Let L denote the Lucas numbers $\{2,1, \dots, l_{i-2} + l_{i-1}, \dots\}$. We consider the sequences L and F as defined and $H_n(L), H_n(F)$ generated by L and F respectively.

$$H_n(L) = \begin{cases} -3 \cdot 2^{n-2} & \text{n even} \\ 2^n & \text{n odd} \end{cases}$$

Proof

Consider the $n \; x \; n$ Matrix M such that $M_{0,0} = M_{(n-1),(n-1)} = 1$, $M_{(n-1),(n-2)} = M_{0,2} = 2$, all other terms on the diagonal from $M_{0,2} \to M_{(n-2),(n-2)}$ and the diagonal $M_{1,0} \to M_{(n-1),(n-2)}$ are 1, and all other terms are 0. In summary, where non-specified ellipses are 0,

$$M_n = \begin{bmatrix} 1 & 2 & 0 & \dots & 0 \\ 1 & 0 & 1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 1 & 0 & 1 \\ 0 & \dots & 0 & 2 & 1 \end{bmatrix}$$

We consider $\det(M)$ in terms of nontrivial permutations. If n is odd, our consideration branches into the collection of permutations beginning with $M_{0,0} = 1$ and the collection beginning with $M_{0,1} = 2$. In each case, there is only one nontrivial path. We conclude for n odd:

$$\det(M_n) = (1 \cdot 1 \cdot \dots \cdot 1 \cdot 2) + (2 \cdot 1 \cdot 1 \cdot \dots \cdot 1)$$

If n is even, there the same logic provides us with

$$\det(M_n) = -(1 \cdot 1 \cdot \dots \cdot 1) + (2 \cdot 1 \cdot 1 \dots \cdot 1 \cdot 2)$$

In general, we can write that $\det(M) = 4$ if n is odd and $\det(M) = 3$ if n is even.

Furthermore,

$$M_nH_n(F) = \begin{bmatrix} 1 & 2 & 0 & \dots & 0 \\ 1 & 0 & 1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 1 & 0 & 1 \\ 0 & \dots & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} f_{n-1} & f_{n-2} & \cdots & f_1 & f_0 \\ f_{n-2} & f_{n-3} & \cdots & f_0 & f_1 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ f_1 & f_0 & \cdots & f_{n-3} & f_{n-2} \\ f_0 & f_1 & \cdots & f_{n-2} & f_{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} f_{n-1}+2f_{n-2} & f_{n-2}+2f_{n-3} & \cdots & f_1+2f_0 & f_0+2f_1 \\ f_{n-1}+f_{n-3} & f_{n-2}+f_{n-4} & \cdots & f_1+f_1 & f_0+f_2 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ f_0+f_2 & f_1+f_1 & \cdots & f_{n-2}+f_{n-4} & f_{n-1}+f_{n-3} \\ 2f_1+f_0 & 2f_0+f_1 & \cdots & 2f_{n-3}+f_{n-2} & 2f_{n-2}+f_{n-1} \end{bmatrix}$$

We can reduce by applying the identity $l_i = f_{i+1} + f_{i-1} = f_i + 2f_{i-1}$ we can reduce

$$= \begin{bmatrix} l_{n-1} & l_{n-2} & \cdots & l_1 & l_0 \\ l_{n-2} & l_{n-3} & \cdots & l_1 & l_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ l_1 & l_0 & \cdots & l_{n-3} & l_{n-2} \\ l_0 & l_1 & \cdots & l_{n-2} & l_{n-1} \end{bmatrix}$$
$$= H_n(L)$$

Therefore, we deduce the equation by multiplicity

$$\begin{aligned} \det(H_n(L)) &= \det(M_nH_n(L)) \\ &= \det(M_n)\det(H_n(L)) \\ &= (-1)^{\frac{(n-1)(n-2)}{2}} 2^{n-2} \det(M_n) \\ &= \begin{cases} (-1)^{(n-1)(n-2)/2} 2^{n-2} (-1)^{n/2} \cdot 3 & \text{n even} \\ (-1)^{(n-1)(n-2)/2} 2^{n-2} (-1)^{(n-1)/2} \cdot 4 & \text{n odd} \end{cases} \\ &= \begin{cases} -3 \cdot 2^{n-2} & \text{n even} \\ 2^n & \text{n odd} \end{cases} \end{aligned}$$

■

Theorem 2.1.3 Arithmetic Sequences and Central Transform

Let $H_n(A)$ be the central Hankel transform of the arithmetic sequence $A = k \cdot \{0,1,2,\dots\}$ for some integer $n \geq n/$ Then

$$\det(H_n(A)) = (-1)^{\frac{(n-1)n}{2}} 2^{n-2} k^n (n-1)$$

Proof

We look at

$$H_n(A) = \begin{bmatrix} (n-1)k & (n-2)k & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ k & 0 & \dots & (n-2)k \\ 0 & k & \dots & (n-1)k \end{bmatrix}$$

By the multiplicity of determinants, we have that the matrix

$$Z = \begin{bmatrix} (n-1) & (n-2) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 1 & 0 & \dots & (n-2) \\ 0 & 1 & \dots & (n-1) \end{bmatrix}$$

Is related to $H_n(A)$ by $\det(H_n(A)) = k^n \det(Z)$. Thus, we can consider Z in isolation to compute the determinant of H; we use elementary column operations on to simplify the matrix without affecting its determinant.

We label the n columns of Z as $col_1, col_2, \dots col_n$. Iteratively, we subtract column col_{i+1} from col_i, in order for each $1 \le i \le n-1$. These operations produce a modified matrix Z'

$$Z' = \begin{bmatrix} -1 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & -1 & \dots & n-2 \\ 1 & -1 & -1 & \dots & n-2 \\ -1 & -1 & -1 & \dots & n-1 \end{bmatrix}$$

Repeating the same elementary column operations for the first $n-2$ columns of the matrix Z'produces a matrix Z''

$$Z'' = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 2 & 0 & 1 \\ 0 & 0 & \iddots & 0 & 0 & 2 \\ 0 & 2 & \dots & 0 & 0 & 3 \\ \iddots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & n-1 \end{bmatrix}$$

Hence, we have that $\det(Z) = \det(Z'')$, and, from above, we see the only non-trivial permutation for the determinant of Z'' is $2^{n-2}(n-1)$. Therefore, we have $\det(Z) = 2^{n-2}(n-1)$, which implies that

$$\det(H_n) = 2^{n-2}k^n(n-1)(-1)^{n(n-1)/2}$$

■

3.2 Fibonacci Identities

Our main proof in this section is that the Fibonacci numbers are equivalently defined by difference between powers of two and the partial sums of the Fibonacci numbers (which will be rigorously defined).

I will present (1) a proof of the theorem that does not rely on determinants, and (2), an incomplete proof that uses determinants. A third proof appears in (Mathes).

The first proof originally "used" the Leibniz formula for the determinant of $H_n(f)$, but on review it was found that the Liebniz formula was merely being danced around in the body of the proof and contributed nothing to the evidence. This is to say that the identity was discovered by looking at determinants, and the proof is motivated by determinants, but determinants do not actually appear in the body of the proof.

Lemma 3.2.1 Intermediate Fibonacci Identity

$$f_n = 2^{n-1} - \sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\binom{n-k}{k+j}$$

Proof:

Consider the following derivation:

$$\begin{aligned}
f_{n+1} &= f_{n+1} \\
&= 2^n - (2^n - f_{n+1}) \\
&= 2^n - \left(2^n - \frac{1}{2}(f_n + f_{n-1} + f_{n+1})\right) \\
&= 2^n - \left(2^n - \frac{1}{2}(f_n + L_n)\right) \\
&= 2^n - \left(2^n - \frac{1}{2}\left(\frac{\varphi^n - \psi^n}{\sqrt{5}} + \varphi^n + \psi^n\right)\right) \\
&= 2^n - \frac{2^{-n}\left((10-4\sqrt{5})(1-\sqrt{5})^n + 5(\sqrt{5}-3)4^n - (\sqrt{5}-5)(1+\sqrt{5})^n\right)}{5(\sqrt{5}-3)}
\end{aligned}$$

Writing

$$\frac{2^{-n}\left((10-4\sqrt{5})(1-\sqrt{5})^n + 5(\sqrt{5}-3)4^n - (\sqrt{5}-5)(1+\sqrt{5})^n\right)}{5(\sqrt{5}-3)} = \varepsilon,$$

But from (Paolo) that the term ε is the recurrence equation describing the relationship

$$a_n = 3a_{n-1} - a_{n-2} - 2a_{n-3};\, a_0 = 0, a_1 = 1, a_2 = 3$$

From (Barry) we have that this function is equivalent to the function

$$\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n-k}{k+j}$$

Thus, shifting algebraically to solve for the typical case f_n, we have the following:

$$f_n = 2^{n-1} - \sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\binom{n-k}{k+j}$$

■

Definition 3.2.1 Fibonacci Partial Sum

Let the function $\mathcal{F}(k,n)$ be defined by

$$\mathcal{F}(k,n) = \sum_{j=0}^{n}\binom{n-j}{j+k}; \qquad \mathcal{F}(0,n) = f_{n+1}$$

We say that $\mathcal{F}(k,n)$ is the k_{th} term of the n_{th} partial sum of the Fibonacci numbers for reasons that will be apparent.

Corollary 3.2.1

We can rephrase Lemma 3.2.1 using this new notation to say that

$$f_n = 2^{n-1} - \sum_{j=0}^{n-1}\mathcal{F}(k,n)$$

Lemma 2: Recurrence relation of $\mathcal{F}(k,n)$

$$\forall k > 0, \qquad \mathcal{F}(k,n) = \sum_{j=0}^{n}\mathcal{F}(k-1,j-1)$$

Assume that up to some z, s $\mathcal{F}(z,n) = \sum_{j=0}^{n}\mathcal{F}(z-1,j-1)$. We apply the definition of $\mathcal{F}(z+1,n)$ and reduce:

$$\begin{aligned}\mathcal{F}(z+1,n) &= \sum_{j=0}^{n}\binom{n-j}{j+z+1}\\ &= \sum_{j=0}^{n}\binom{n-j+1}{j+z+1} - \sum_{j=0}^{n}\binom{n-j}{j+z}\end{aligned}$$

By assumption and definition respectively,

$$\begin{aligned}\mathcal{F}(z+1,n+1) - \mathcal{F}(z,n) &= -\sum_{j=0}^{n}\sum_{u=0}^{j-1}\binom{j-u-1}{u+z-1} + \sum_{j=0}^{n}\binom{n+1-j}{j+z+1}\\ &= \sum_{j=0}^{n}\left(\sum_{u=0}^{j}\binom{j-u-1}{u+z-1} - \sum_{u=0}^{n-j}\binom{u}{j+z}\right)\\ &= \sum_{j=0}^{n}\left(\sum_{u=0}^{j}\left(\binom{j-1-u}{z-1+u} - \binom{u}{n+z-j}\right)\right)\\ &= \sum_{j=0}^{n}\sum_{u=0}^{j}\binom{j-u-1}{u+z}\end{aligned}$$

Thus, by induction,

$$\mathcal{F}(k,n) = \sum_{j=0}^{n} \mathcal{F}(k-1, j-1) \quad \forall k > 0$$

■

Theorem 3.2.1 New Fibonacci Identity

$$f_{n+2} = 2^{n+1} - \sum_{k=0}^{n} \mathcal{F}(k,n)$$

Where $\mathcal{F}(k,n)$ is the k_{th} term of the recursive sum of the Fibonacci numbers.

Proof (1):

From Corollary 3.2.1,

$$f_n = 2^{n-1} - \sum_{j=0}^{n-1} \mathcal{F}(k,n)$$

From Lemma 3.2.2, however, we know this to be the recursive sum of sets of Fibonacci numbers. This completes the proof.

■

Example 3.2.1 Partial Sums Identity

A table of the first few partial sums is given by

Table 2 Partial Sums of Fibonacci Numbers

	$k=0$	1	2	3	4	5	6	7	8
$n=0$	0	1	1	2	3	5	8	13	21
1	0	0	1	2	4	7	12	20	33
2	0	0	0	1	3	7	14	26	46
3	0	0	0	0	1	4	11	25	51
4	0	0	0	0	0	1	5	16	41

We look to the columns of the above table; the following can be verified.

$$2^{3-2} = f_{3-1} + (1) \rightarrow 2 = 1 + 1$$
$$2^{4-2} = f_{4-1} + (1+1) \rightarrow 4 = 2 + 2$$
$$2^{5-2} = f_{5-1} + (2+2+1) \rightarrow 8 = 3 + 5$$
$$2^{6-2} = f_{6-1} + (3+4+3+1) \rightarrow 16 = 5 + 11$$
$$2^{7-2} = f_{7-1} + (5+7+7+4+1) \rightarrow 32 = 8 + 24$$

■

Proof (2) of Theorem 3.2.1

This proof is incomplete. I outline it because it makes use of determinants and I believe that it is close to being completed[12]. The idea is to iteratively sum the columns of the Toeplitz matrix of f until the partial sums appear in the top row (call the new matrix G). We compare this equivalent matrix to a matrix with the partial sums along the top row such that its determinant is the sum of those terms (call this matrix U); the factor that links them, i.e., the L such that $LG = U$, must have a determinant of 1 if this theorem is true. Since G has the same determinant as $T(f)$, the proof would be complete in that case. Showing that the factor is unimodular is, of course, the difficult part of the proof, and has evaded me in entirety (though L does have some nice demonstrable recursive properties that make me think that the proof is possible).

Theorem 3.2.2

$$\sum_{\sigma}\prod_{i=0}^{n-1}\mathrm{sgn}_{\sigma} f_{|i-\sigma_i|} = (-1)^{n-1}2^{n-2}$$

This theorem is a direct consequence of Theorem 2.1.1, by applying the Leibniz formula for determinant.

3.3: Injectivity of the Central Hankel

Theorem 3.3.1 The Central Transform is injective on the Fibonacci numbers

Let $H_n(S)$ be the n-dimensional Central Hankel transform of a sequence S.

We show that if $H_n(S) = H_n(f)$ where S is an arbitrary sequence and f is the Fibonacci numbers, then $S = f$. We perform this proof by induction on the Toeplitz matrix $T_n(f)$, recalling that, in magnitude of determinant, this is the same consideration.

The base case is easy and requires up to $n = 3$ to be computed manually. In the one dimensional case, $[x_0] = 0 \to x_0 = 0$. In the two dimensional case $\begin{bmatrix} 0 & x_1 \\ x_1 & 0 \end{bmatrix} = 1$ which means that $x_1 = \pm 1$.

We now divide this proof into two cases, (A) where $x_1 = 1$ and (B) where $x_1 = -1$. We show that when $x_1 = 1$, S is the Fibonacci numbers going forwards $(0,1,1,2,\ldots)$ and when $x_1 = 1$, S is the Fibonacci numbers going backwards $(0,-1,1,-2,\ldots)$. First, assume that $x_1 = 1$.

The next dimension can be evaluated as

$$\begin{aligned} 2 &= \begin{bmatrix} 0 & 1 & x_2 \\ 1 & 0 & 1 \\ x_2 & 1 & 0 \end{bmatrix} \\ 2 &= 0 - 1\begin{bmatrix} 0 & x_2 \\ x_2 & 0 \end{bmatrix} + x_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ 0 &= x_2^2 - x_2 + 2 \\ 1 &= x_2 \end{aligned}$$

The inductive step finishes this case of the proof.

[12] Or is hopeless and impossible; I'm on the fence.

Let $T_n(S)$ be defined by a sequence $S = \{f_0, f_1, \dots, f_{n-2}, x\}$ where x is unknown, and let $\det(T_n) = (-1)^n 2^{n-2}$.

$$T_n = \begin{bmatrix} 0 & 1 & 1 & 2 & 3 & \dots & x \\ 1 & 0 & 1 & 1 & 2 & \dots & \vdots \\ 1 & 1 & 0 & 1 & 1 & \dots & 3 \\ 2 & 1 & 1 & 0 & 1 & \dots & 2 \\ 3 & 2 & 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ x & \dots & 3 & 2 & 1 & 1 & 0 \end{bmatrix}$$

Subtract $col_{n-3} + col_{n-2}$ into col_{n-1}, and then subtract $row_{n-3} + row_{n-2}$ into row_{n-1}, then scale the last column by $\frac{1}{2}$. We have , letting $x' = x - (f_{n-3} + f_{n-2}) = x - f_{n-1}$

$$T_n' = \begin{bmatrix} 0 & 1 & 1 & 2 & 3 & \dots & x'/2 \\ 1 & 0 & 1 & 1 & 2 & \dots & \vdots \\ 1 & 1 & 0 & 1 & 1 & \dots & 0 \\ 2 & 1 & 1 & 0 & 1 & \dots & 0 \\ 3 & 2 & 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ x' & \dots & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

We have zeroed out the last row and last column, *except* for the first element of both, which is x', and the term in the bottom right of the matrix, which is -1. Since we scaled the final column by $1/2$ (meaning that $\det(T_n')$, as written equals $\frac{1}{2}\det(T_n)$).

Let

$$\begin{aligned} A &= T_{n-1} \\ b &= [x'/2, 0, \dots, 0] \\ c &= [x', 0, \dots, 0]^T \\ d &= -1 \end{aligned}$$

We have that

$$\det\left(\begin{bmatrix} A & b \\ c & d \end{bmatrix}\right) = (d-1)\det(A) + \det(A - BC)$$

By substitution,

$$\det(T_n) = (2)\det(T_{n-1}) + \det(T_{n-1} - bc)$$

The product bc is given by, letting $z = -\frac{1}{2}x'^2$

$$bc = \begin{bmatrix} -z & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, $T_{n-1} - bc$ is given by

$$T_{n-1} - bc = \begin{bmatrix} z & 1 & 1 & 2 & 3 & \dots & f_{n-2} \\ 1 & 0 & 1 & 1 & 2 & \dots & \vdots \\ 1 & 1 & 0 & 1 & 1 & \dots & 3 \\ 2 & 1 & 1 & 0 & 1 & \dots & 2 \\ 3 & 2 & 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ f_{n-2} & \dots & 3 & 2 & 1 & 1 & 0 \end{bmatrix}$$

We can now subtract $row_{n-3} + row_{n-2}$ from row_{n-1}, as before, and yielding

$$\det(T_{n-1} - bc) = \det\begin{bmatrix} z & 1 & 1 & 2 & 3 & \dots & f_{n-2} \\ 1 & 0 & 1 & 1 & 2 & \dots & \vdots \\ 1 & 1 & 0 & 1 & 1 & \dots & 3 \\ 2 & 1 & 1 & 0 & 1 & \dots & 2 \\ 3 & 2 & 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & \dots & 0 & 0 & 0 & 0 & -2 \end{bmatrix} = G_{n-1}$$

Since G_{n-1} has the same final three columns as $H_{n-1}(f)$, we can recursively compute its determinant as before, finding that $\det(G_k) = -2\det(G_{k-1})$. This relation holds until we have

$$G_3 = \begin{bmatrix} z & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$
$$\det G_3 = z\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix}$$

As the last two cofactors cancel, we have

$$\det G_3 = z(1) = z$$
$$= -\frac{1}{2}x'^2 = -\frac{1}{2}(x - f_{n-1})^2$$

So

$$\det G_{n-1} = -2^{n-3}\frac{1}{2}(x - f_{n-1})^2$$

Putting together our results, we have

$$2^{n-2} = \det(T_n)$$
$$2^{n-2} = (2)\det(T_{n-1}) + \det(T_{n-1} - bc)$$
$$2^{n-2} = 2(2^{n-3}) - \left(2^{n-3}\frac{1}{2}(x - f_{n-1}^2)\right)$$
$$0 = (x - f_{n-1})$$
$$x = f_{n-1}$$

When we assigned G_{n-1}, we made the assumption that $\dim G_{n-1} \geq 3$. This is consistent with our base case treatment.

On the other hand, let $x_1 = -1$. The exact same proof as above can be repeated with the negative Fibonacci numbers.

3.4 Closed Form for Central Hankel Transform of Fibonacci-Recurrent Sequences

Theorem 3.4.1

Where S is a Fibonacci-Recurrent sequence with initial values[13] $s_0 \neq 0, s_1$, the central Hankel transform of S is given by

$$\det\big(H(s)\big) = s_0^{n-1} s_1 [G_{n-2} - 2G_{n-3} + G_{n-4}]$$

Where

$$G_n = c_1 A^n + c_2 B^n$$

For A, B, defined as constants in terms of of s_0 and s_1 by $A = \left(\frac{D+\sqrt{D^2-4}}{2}\right)^{n+1}$, $B = \left(\frac{D-\sqrt{D^2-4}}{2}\right)^{n+1}$ when $D = \frac{2s_1 - s_0}{s_0}$ and $c_1 = \frac{1}{\sqrt{D^2-4}}, c_2 = -c_1$

Proof

Let $f = \{f_0, f_1, \dots\} = \{0,1,1,\dots\}$, be the Fibonacci sequence. Let $S = \{s_0, s_1, \dots, s_{n-2}, s_{n-1}\}$ where $s_k = s_{k-2} + s_{k-1}$ be a general Fibonacci-recurrent sequence. As usual, let The Central Hankel matrix generated by S is given by

$$H_n(S) = \begin{bmatrix} s_{n-1} & \cdots & s_1 & s_0 \\ \vdots & ⋰ & ⋰ & s_1 \\ s_1 & s_0 & ⋰ & \vdots \\ s_0 & s_1 & \cdots & s_{n-1} \end{bmatrix}$$

Definition 4.1.1: (tri-diagonal) L-Factor

Where $g = 2s_1 - s_0$,

$$L_n(S) = \frac{1}{2} \begin{bmatrix} 2s_1 & s_0 & & & \\ 2s_0 & g & \ddots & & \\ 0 & s_0 & \ddots & s_0 & 0 \\ & & \ddots & g & 2s_0 \\ & & & s_0 & 2s_1 \end{bmatrix}$$

Theorem 4.1.1: $\boldsymbol{H_n(f) \cdot L_n(s) = H_n(s)}$

Proof: The proof is given at the appendix of this section for readability; it is just multiplication. ∎

Theorem 4.1.2: $\boldsymbol{det\big(H_n(s)\big) = det\big(L_n'(s)\big)}$ where $\boldsymbol{L_n'(s)}$ is $\boldsymbol{L_n(s)}$ without the scalars of 2 in the first and last column

We have now shown that

[13] When $s_0 = 0$ we have the k-Fibonacci numbers which are handled easily by a separate case, but are not of interest here; this certainly is not inhibiting.

$$\begin{bmatrix} f_{n-1} & f_{n-2} & \cdots & f_1 & f_0 \\ f_{n-2} & f_{n-3} & \cdots & f_0 & f_1 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ f_1 & f_0 & \cdots & f_{n-3} & f_{n-2} \\ f_0 & f_1 & \cdots & f_{n-2} & f_{n-1} \end{bmatrix} \begin{bmatrix} 2s_1 & s_0 & & & \\ 2s_0 & g & \ddots & & \\ 0 & s_0 & \ddots & s_0 & 0 \\ & & \ddots & g & 2s_0 \\ & & & s_0 & 2s_1 \end{bmatrix} = 2H_n(S)$$

This allows us to evaluate the determinant of $H_n(s)$ by multiplicity:

$$\det(H_n(f)L_n) = \det\big(2H_n(s)\big)$$
$$\det\big(H_n(f)\big)\det(L_n) = \det\big(2H_n(s)\big)$$

Let L' equal L with the first column and last column scaled by $\frac{1}{2}$ so that $\det(L') = \frac{1}{4}\det L$, or equivalently $\det(L_n) = 4\det(L'_n)$. It follows that

$$\det\big(H_n(f)\big)\left(4\det\left(\frac{1}{2}L'_n\right)\right) = \det(\,H_n(s))$$
$$4\big[\det\big(H_n(f)\big)\cdot 2^{-n}\det(L'_n)\big] = \det\big(H_n(s)\big)$$
$$4\,2^{n-2}\cdot\, 2^{-n}\det(L'_n) = H_n(s)$$
$$H_n(S) = \det(L'_n)$$

■

Theorem 4.1.1 General Fibonacci Determinant (Continued)

By definition,

$$\det\big(H_n(S)\big) = \det\begin{bmatrix} s_5 & s_4 & s_3 & s_2 & s_1 & s_0 \\ s_4 & s_3 & s_2 & s_1 & s_0 & s_1 \\ s_3 & s_2 & s_1 & s_0 & s_1 & s_2 \\ s_2 & s_1 & s_0 & s_1 & s_2 & s_3 \\ s_1 & s_0 & s_1 & s_2 & s_3 & s_4 \\ s_0 & s_1 & s_2 & s_3 & s_4 & s_4 \end{bmatrix}$$

Which is by **Theorem 1** equivalent to the L' factor written by, with $g = 2s_1 - s_0$

$$\det\big(L'_n(S)\big) = \det\begin{bmatrix} s_1 & s_0 & 0 & 0 & 0 & 0 \\ s_0 & g & s_0 & 0 & 0 & 0 \\ 0 & s_0 & g & s_0 & 0 & 0 \\ 0 & 0 & s_0 & g & s_0 & 0 \\ 0 & 0 & 0 & s_0 & g & s_0 \\ 0 & 0 & 0 & 0 & s_0 & s_1 \end{bmatrix}$$

Which is equivalent to the following, by scaling L'_n by $1/s_0$, where $p = s_1/s_0$ and $D = g/s_0$

$$= s_0^n\det\begin{bmatrix} p & 1 & 0 & 0 & 0 & 0 \\ 1 & D & 1 & 0 & 0 & 0 \\ 0 & 1 & D & 1 & 0 & 0 \\ 0 & 0 & 1 & D & 1 & 0 \\ 0 & 0 & 0 & 1 & D & 1 \\ 0 & 0 & 0 & 0 & 1 & p \end{bmatrix}$$

Define the following matrix to be used in consideration:

$$\mathcal{G}_n = \begin{bmatrix} D & 1 & 0 & 0 & 0 & 0 \\ 1 & D & 1 & 0 & 0 & 0 \\ 0 & 1 & D & 1 & 0 & 0 \\ 0 & 0 & 1 & D & 1 & 0 \\ 0 & 0 & 0 & 1 & D & 1 \\ 0 & 0 & 0 & 0 & 1 & D \end{bmatrix}_n, \quad G_n = \det(\mathcal{G}_n)$$

We explicitly evaluate this determinant by the following derivation:

$$= (s_0^n)\left(p \det \begin{bmatrix} D & 1 & 0 & 0 & 0 & 0 \\ 1 & D & 1 & 0 & 0 & 0 \\ 0 & 1 & D & 1 & 0 & 0 \\ 0 & 0 & 1 & D & 1 & 0 \\ 0 & 0 & 0 & 1 & D & 1 \\ 0 & 0 & 0 & 0 & 1 & s \end{bmatrix}_{n-1} - \det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & D & 1 & 0 & 0 & 0 \\ 0 & 1 & D & 1 & 0 & 0 \\ 0 & 0 & 1 & D & 1 & 0 \\ 0 & 0 & 0 & 1 & D & 1 \\ 0 & 0 & 0 & 0 & 1 & p \end{bmatrix}_{n-1} \right)$$

$$= (s_0^n)\left(p \det \begin{bmatrix} D & 1 & 0 & 0 & 0 & 0 \\ 1 & D & 1 & 0 & 0 & 0 \\ 0 & 1 & D & 1 & 0 & 0 \\ 0 & 0 & 1 & D & 1 & 0 \\ 0 & 0 & 0 & 1 & D & 1 \\ 0 & 0 & 0 & 0 & 1 & p \end{bmatrix}_{n-1} - \det \begin{bmatrix} D & 1 & 0 & 0 & 0 & 0 \\ 1 & D & 1 & 0 & 0 & 0 \\ 0 & 1 & D & 1 & 0 & 0 \\ 0 & 0 & 1 & D & 1 & 0 \\ 0 & 0 & 0 & 1 & D & 1 \\ 0 & 0 & 0 & 0 & 1 & p \end{bmatrix}_{n-1} \right)$$

$$= (s_0^n)\left(s \left(G_{n-2} - \det \begin{bmatrix} D & 1 & 0 & 0 & 0 & 0 \\ 1 & D & 1 & 0 & 0 & 0 \\ 0 & 1 & D & 1 & 0 & 0 \\ 0 & 0 & 1 & D & 1 & 0 \\ 0 & 0 & 0 & 1 & D & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{n-2} \right) - \det \begin{bmatrix} D & 1 & 0 & 0 & 0 & 0 \\ 1 & D & 1 & 0 & 0 & 0 \\ 0 & 1 & D & 1 & 0 & 0 \\ 0 & 0 & 1 & D & 1 & 0 \\ 0 & 0 & 0 & 1 & D & 1 \\ 0 & 0 & 0 & 0 & 1 & s \end{bmatrix}_{n-1} \right)$$

$$= (s_0^n)\left(p(G_{n-2} - G_{n-3}) - \det \begin{bmatrix} D & 1 & 0 & 0 & 0 & 0 \\ 1 & D & 1 & 0 & 0 & 0 \\ 0 & 1 & D & 1 & 0 & 0 \\ 0 & 0 & 1 & D & 1 & 0 \\ 0 & 0 & 0 & 1 & D & 1 \\ 0 & 0 & 0 & 0 & 1 & p \end{bmatrix}_{n-1} \right)$$

$$= (s_0^n)\big(p(G_{n-2} - G_{n-3}) - (pG_{n-3} - G_{n-4})\big)$$

We have,

$$= s_0^{n-1} s_1 [G_{n-2} - 2G_{n-3} + G_{n-4}]$$

Since $\mathcal{G}$ is tri-diagonal and Toeplitz, we can apply (Cinkir) to obtain a closed form for its determinant.[14]

[14] Before discovering (Cinkir), I did this computation by hand by considering the continuant of $\mathcal{G}$, any recurrence relation has a characteristic polynomial the roots of which indicate the bases of exponentials that solve the function. Think of it as solving a differential equation and discovering that you got powers of e, logically. This explains the apparition of the discriminant of the quadratic formula as the base of the exponential. (Cinkir) uses Cauchy-Benet. Note that over real numbers, this tells us that the determinant only has a closed form when $D^2 \geq 4$ or else the characteristic polynomial of the continuant has no real roots, though over complex numbers this is irrelevant. For a great overview of continuants in tri-diagonal determinant computation, old-school is best (Muir)

$$G_n = \frac{1}{\sqrt{D^2-4}}\left[\left(\frac{D+\sqrt{D^2-4}}{2}\right)^{n+1} - \left(\frac{D-\sqrt{D^2-4}}{2}\right)^{n+1}\right]$$

Let $A = \left(\frac{D+\sqrt{D^2-4}}{2}\right)^{n+1}$, $B = \left(\frac{D-\sqrt{D^2-4}}{2}\right)^{n+1}$, $c_1 = \frac{1}{\sqrt{D^2-4}}$, $c_2 = -c_1$, we have

$$G_n = c_1 A^{n+1} + c_2 B^{n+1}$$

■

4.A (Appendix)

Recall:

$$L_n(S) = \frac{1}{2}\begin{bmatrix} 2s_1 & s_0 & & & \\ 2s_0 & g & \ddots & & \\ 0 & s_0 & \ddots & s_0 & 0 \\ & & \ddots & g & 2s_0 \\ & & & s_0 & 2s_1 \end{bmatrix},\ H_n(S) = \begin{bmatrix} s_{n-1} & \cdots & s_1 & s_0 \\ \vdots & ⋰ & ⋰ & s_1 \\ s_1 & s_0 & ⋰ & \vdots \\ s_0 & s_1 & \cdots & s_{n-1} \end{bmatrix}$$

Proof of Theorem 4.1.1: $\boldsymbol{H_n(f)L_n(S) = H_n(L)}$

This theorem can be proven by multiplying $H_n\ (f)$ by $L_n(s)$ (hereafter denoted L_n when unambiguous). The top right corner of $H_n(f)L_n$ is defined by

$$\begin{aligned}
&\frac{1}{2}\begin{bmatrix} 2(f_{n-1}s_1 + f_{n-2}s_0) & f_{n-1}s_0 + f_{n-2}(2s_1 - s_0) + f_{n-3}s_0 \\ 2(f_{n-2}s_1 + f_{n-3}s_0) & f_{n-2}s_0 + f_{n-3}(2s_1 - s_0) + f_{n-4}s_0 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 2s_{n-1} & f_{n-1}s_0 + 2s_1 f_{n-2} - s_0 f_{n-2} + f_{n-3}s_0 \\ 2s_{n-2} & f_{n-2}s_0 + 2s_1 f_{n-3} - s_0 f_{n-3} + f_{n-4}s_0 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 2s_{n-1} & s_0([f_{n-2} + f_{n-3}] - f_{n-2} + f_{n-3}) + 2s_1(f_{n-2}) \\ 2s_{n-2} & s_0([f_{n-3} + f_{n-4}] - f_{n-3} + f_{n-4}) + 2s_1(f_{n-3}) \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 2s_{n-1} & s_0(2f_{n-3}) + 2s_1(f_{n-2}) \\ 2s_{n-2} & s_0(2f_{n-4}) + 2s_1(f_{n-3}) \end{bmatrix} \\
&= \begin{bmatrix} s_{n-1} & s_{n-2} \\ s_{n-2} & s_{n-3} \end{bmatrix}
\end{aligned}$$

By almost the exact same derivation, the bottom right corner is given by

$$\frac{1}{2}\begin{bmatrix} f_{n-2}s_0 + f_{n-3}(2s_1 - s_0) + f_{n-4}s_0 & f_{n-1}s_0 + f_{n-2}(2s_1 - s_0) + f_{n-3}s_0 \\ 2(f_{n-2}s_1 + f_{n-3}s_0) & 2(f_{n-1}s_1 + f_{n-2}s_0) \end{bmatrix} = \begin{bmatrix} s_{n-3} & s_{n-2} \\ s_{n-2} & s_{n-1} \end{bmatrix}$$

Now consider some term that does not lie in either the bottom right or top left corner:

$$[H_n(f)L_n]_{i,j} = \sum_{k=0}^{n} (H_n(f)_{k,0}(L_{n\,0,k})$$

However, only three terms of a given column of L_n are non-zero, specifically $k \in [j-1, j, j+1]$. Elliminating all other terms,

$$[H_n(f)L_n]_{i,j} = \left(H_n(f)_{j-1,0}\right)\left(L_{n_{j,j-1}}\right) + \left(H_n(f)_{j,0}\right)\left(L_{n_{j,j}}\right) + \left(H_n(f)_{j,0}\right)\left(L_{n_{j,j}}\right)$$

The terms of L_n are known to be s_0, g, s_0 respectively, so

$$[H_n(f)L_n]_{i,j} = \left(H_n(f)_{j-1,0}\right)s_0 + \left(H_n(f)_{j,0}\right)g + \left(H_n(f)_{j,0}\right)s_0$$

But $H_n(f)_{i,j} = f_{|i-j|}$ by definition, s

$$\begin{aligned} 2[H_n(f)L_n]_{i,j} &= f_{|i-j+1|}s_0 + f_{|i-j|}g + f_{|i-j+1|}s_0 \\ &= f_{|i-j+1|}s_0 + f_{|i-j|}g + s_0\left(f_{|i-j|} + f_{|i-j-1|}\right) \\ &= s_0\left(2f_{|i-j-1|} + f_{|i-j|}\right) + f_{|i-j|}(2s_1 - s_0) \\ &= 2\left(s_0\left(f_{|i-j-1|}\right) + s_1 f_{|i-j|}\right) \\ &= 2s_{|i-j|} \end{aligned}$$

■

Lemma 2.4.2 Proof : $s_0 f_{n-1} + s_1 f_n = s_n$ for all sequences $s_n = s_{n-1} + s_{n-2}$

I assume that this has been proven before, but I do not see it in any of my references and it is faster to reprove than to look up, so here is a simple inductive proof. Note that when $n = 1$, we have

$$s_0 f_0 + s_1 f_1 = s_1$$

Assume that when $n \leq k$,

$$s_0 f_{n-1} + s_1 f_n = s_n$$

We show that it is true when $n = k + 1$

$$\begin{aligned} &s_0 f_{k+1-1} + s_1 f_{k+1} \\ &= s_0 f_k + s_1 f_{k+1} \\ &= s_0 f_k + s_1(f_k + f_{k-1}) \\ &= s_0 f_k + s_1 f_k + s_1 f_{k-1} \\ &= (s_1 f_{k-1} + s_0 f_k) + s_1 f_k \\ &= (s_0 f_{k-2} + s_1 f_{k-1}) + (s_0 f_{k-1} + s_1 f_k) \end{aligned}$$

By assumption

$$\begin{aligned} &= s_{k-1} + s_k \\ &= s_{k+1} \end{aligned}$$

Example 3.4.1 General Fibonacci Exposition

Consider $s_0 = 11$, $s_1 = 72$, $s_n = s_{n-1} = s_{n-2}$. Theorem 3.4.1 allows us to evaluate its determinant. Let $n = 6$ for exposition. As in the derivation of Theorem 3.4.1, the determinants of the following matrices are equal[15]

[15] Except that there's round-off in the way I've displayed them

$$T_6(s) = \begin{bmatrix} 11 & 72 & 83 & 155 & 238 & 393 \\ 72 & 11 & 72 & 83 & 155 & 238 \\ 83 & 72 & 11 & 72 & 83 & 155 \\ 155 & 83 & 72 & 11 & 72 & 83 \\ 238 & 155 & 83 & 72 & 11 & 72 \\ 393 & 238 & 155 & 83 & 72 & 11 \end{bmatrix}$$

$$L_6'(s) = \begin{bmatrix} 72 & 11 & & & & \\ 11 & 133 & 11 & & & \\ & 11 & 133 & 11 & & \\ & & 11 & 133 & 11 & \\ & & & 11 & 133 & 11 \\ & & & & 11 & 72 \end{bmatrix}$$

$$L_6''(s) = \begin{bmatrix} 6.55 & 1 & & & & \\ 1 & 12.1 & 1 & & & \\ & 1 & 12.1 & 1 & & \\ & & 1 & 12.1 & 1 & \\ & & & 1 & 12.1 & 1 \\ & & & & 1 & 6.55 \end{bmatrix}$$

The $\mathcal{G}$ matrix defined in the derivation of Theorem 3.4.1 is given by

$$\mathcal{G}_6(s) = \begin{bmatrix} 12.1 & 1 & & & & \\ 1 & 12.1 & 1 & & & \\ & 1 & 12.1 & 1 & & \\ & & 1 & 12.1 & 1 & \\ & & & 1 & 12.1 & 1 \\ & & & & 1 & 12.1 \end{bmatrix}$$

Example 3.5.1 Lucas Numbers Revisited

The equation for the Lucas numbers can be re-derived by applying the general formula. While not difficult, this, unfortunately, is a slightly laborious task because the discriminant of the continuant is imaginary. The easiest approach is to use Euler's formula and manipulate trigonometric functions in the algebraic step, but this is not relevant to our discussion so we omit the full derivation.

4. Hankel Operators

> *We apologize for the fact that in the title of the Tensors talk in the last newsletter, the words "theoretical physics" came out as "impossible ideas"* - Archimedeans' Newsletter, January 1986[16]

We claim the discoveries we have made are motivated by a parallel to a rich community of results in Hankel theory. Therefore, we give an overview of this branch of operator theory.

In this section, we discuss Hankel Matrices as operators and step enumerate the process in which one finds Hankel Operators embedded in Hilbert and Hardy spaces. This section constitutes discussion of the

[16] This is quoted in (Partington), but is meant here as a quip directed toward Kelvin Lui's excellent thesis this year on Tensor products

rigorous theoretical component behind Hankel Operators, as opposed to the significant applied material available.

Hankel Operators play no surface-level role in the research I helped conduct, so I will not belabor the proofs of the following statements; appropriate references will be made to the location of all claims. I will try and keep those references diverse.

Generally, in this section I use "lemma" to refer to a statement with a proof that is outsourced to another text. I will reserve the word "theorem" for when the proof is in-lined.

4.0 Preliminary Definitions (Vector Spaces)

Background

The reader is assumed, in this section, to have knowledge of standard linear algebra theorems and definitions. If an orthonormal basis or the notion of linearity is intimidating, perhaps see (Brescher) or (Vijicic).

We recall that a Vector Space is a set with addition, scalar multiplication and various rules –additive and multiplicative associativity, additive commutativity, a neutral, negative, and identity element, and distributive qualities (Brescher). If not otherwise defined, $\mathbb{V}$ will denote an arbitrary vector space.

Definition 4.0.1 Inner Product (Brescher)

For $x, y, z \in \mathbb{V}$ and $c \in \mathbb{R}$ the *inner product* of x, y, denoted $\langle x, y \rangle$, is a map $\mathbb{V}^2 \to \mathbb{R}$ that is symmetric, linear in x, and positive definite. Exactly

i. $\langle x, y \rangle = \langle y, x \rangle$
ii. $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$
iii. $\langle cx, y \rangle = c\langle x, y \rangle$
iv. $\langle x, x \rangle > 0$

Definition 4.0.2 Norm (Brescher)

The norm of $x \in \mathbb{V}$ is denoted $\|x\|$ and defined to be

$$\|x\| = \sqrt{\langle x, x \rangle}$$

■

Authors like (Partington) and (Pohl) define these in the opposite order, $\|\cdot\|$ leads to $\langle \cdot,\cdot \rangle$, not $\langle \cdot,\cdot \rangle$ leads to $\|\cdot\|$. The order does not matter, but we prefer (Brescher)'s pedagogy. In any case, it is clear that for a particular vector space, defining a particular norm function or inner product function is sufficient to imply the latter.

Definition 4.0.3 Normed/Inner Product Space

A vector space with a norm is a normed space; a vector space with an inner product is an inner product space. The definitions are tautological.

Example 4.0.1 Normed Spaces (Partington)

Two particular normed spaces will be recurrent in our analysis:

- Let $\boldsymbol{\ell_p}$ be the space of sequences $\{x_i\}$ such that

$$\|(x_i)\| = \left(\sum_{1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}$$

  One can verify (or see (Brescher) 5.5.2) that the norm and associated inner product satisfy the Definitions 1.0.2/3. Of particular interest is ℓ_2, the set of "square summable" sequences.
- Let $\boldsymbol{L_p(X)}$ denote the set of functions, f, over an interval X where[17]

$$\|f\| = \left(\int_X |f(t)|^p \, dt\right)^{\frac{1}{p}}$$

[17] First of all, X can be infinite, though we tend to think of things in finite terms when possible. Secondly, when the following condition is met, the function is called "Lebesgue integrable". If these spaces are interesting to you, see (Gordon), e.g., for their derivation and analysis

4.1 Hilbert Spaces

In this section, we strive not to describe the fundamental theorems regarding Hilbert-space theory, but to build up a useful set of tools for discussing Hankel operators on a Hilbert space.

Notation

When we consider operators, we consider continuous linear operators on vector space with an inner product, or "Hilbert" space, over $\mathbb{C}$. A Hilbert space will be denoted $\mathcal{H}$. The set of linear functions over vector spaces $\mathcal{V}$ and $\mathcal{W}$ is denoted $\mathcal{L}(\mathcal{V},\mathcal{W})$. When unspecified, assume that $A \in \mathcal{L}(\mathcal{H},\mathcal{H})$. We use $\mathbb{R}$ to denote the real numbers, $\mathbb{C}$ to denote the complex numbers, and $\mathbb{F}$ when we want either the real or complex numbers but do not want to specify. If not defined otherwise, V and W are typically subspaces of $\mathbb{F}$. When not defined otherwise, I denotes the identity operator.

I formally apologize that the words "Hilbert", "Hardy", and "Hankel" all start with an "H", which may make notation confusing in later sections. I suspect that this is the primary limiting factor on research in this area.

Definition 4.1.1 Norm: The norm[18] of A is given by the symbol $\|\cdot\|$ and is defined to be

$$\|A\| = \sup\left(\left\{\frac{\|Ax\|}{\|x\|} \mid x \in \mathbb{C}/\emptyset\right\}\right)$$

Definition 4.1.2: Adjoint (Axler)**:** The *adjoint* of $T \in \mathcal{L}(V,W)$ is the function A^* where $A^*: W \to V$ such that[19]

$$\langle Tv, w\rangle = \langle v, T^*w\rangle$$

Lemma 4.1.1 Properties of Adjoint (Axler)

For every $v \in V$ and every $w \in W$, we note that the adjoint preserves linearity, and that the following are true where $U \subset \mathbb{F}$:

$$\begin{aligned}
&(S+T)^* = S^* + T^* \ \ \forall S,T \in \mathcal{L}(V,W)\\
&(\lambda T)^* = \bar{\lambda}T^* \ \ \forall \lambda \in \mathbb{F}, T \in \mathbb{L}(V,W)\\
&(T^*)^* = T \ \forall T \in \mathcal{L}(V,W)\\
&I^* = I\\
&(\mathrm{ST})^* = T^*S^* \ \ \forall T \in \mathcal{L}(V,W), S \in \mathcal{L}(W,U)
\end{aligned}$$

Let $\hat{B}$ denote the "unit ball" such that $\hat{B} = \{x \mid \|x\| < 1\}$ for all $x \in \mathcal{H}$

Definition 4.1.3 Compact Set: (Abadir)**:** A collection $\mathcal{A}$ of sets is said to "cover" a set $\mathcal{B}$ when

$$\mathcal{B} \subset \bigcup_{a \in \mathcal{A}} a$$

---

[18] For a more general discussion of norms on a generalized complex Vector space see (Pohl), e.g. or the preceding section. Given is the norm for a linear function from a Hilbert space to a Hilbert space, as A is defined

[19] $\langle\cdot,\cdot\rangle$ denotes the inner product. (Pohl) has a good brief summary of these too

A *compact* set $\mathcal{B}$ is a set of elements of $\mathbb{F}$ such that each collection $\mathcal{A}$ of open sets covering $\mathcal{B}$ has a finite subcollection able to cover $\mathcal{B}$.

Definition 4.1.4 Compact Operator: (Partington) We say that A is *compact* exactly when the closure of $A(\hat{B})$ is compact.

Lemma 4.1.2: Compactness of Operators:

The following are equivalent:

1. A is compact
2. (Horn) A is closed and bounded, i.e., the limit of any norm-convergent sequence from A is in A, and A is contained in a ball of finite radius.
3. (Partington) Ax has a convergent subsequence if x is a bounded sequence
4. A^* is compact
5. αA is compact

Definition 4.1.4 Self-adjoint Operator

We say that A is *self-adjoint*[20] if $A^* = A$. Some interesting properties of self-adjoint operators can be found in (Axler), particularly with relation to the Spectral Theorem.[21]

Lemma 4.1.3 Compact Self-Adjoint Operator (Partington)

A is a compact self-adjoint operator if and only if there exist a sequence λ_n of real numbers which tends to zero, and an orthonormal basis x_n of $\mathcal{H}$ such that[22]

$$A\beta = \sum_1^\infty \lambda_n \langle x, x_n \rangle x_n$$

■

We note from Lemma 1.1.3 that the basis x_n also constituted the eigenvectors of A where λ_n corresponded to the eigenvalues. Hence the use of the notation of x for a basis where β would have been initially more natural. By the spectral theorem (Berberian), A is diagonal with respect to x.

Lemma 4.1.4 Schmidt Expansion:

An operator T is compact if and only if there exist orthonormal sequences v_i, w_i for $i \geq 1$ and decreasing scalars $\sigma_i \to 0$ such that

$$Tx = \sum_1^\infty \sigma_i \langle x, v_i \rangle w_i$$

---

[20] Or "Hermitian," if you want to be fancy (Partington)

[21] (Axler) is not always dealing with Hilbert spaces, but he *is* dealing with linear operators on a vector space, and almost all of his conclusions are relevant, and can be much easier to follow than the more general versions found in other texts

[22] This statement is a little trite; one might observe that we've defined a compact Hermitian operator without defining a Hermitian operator. For a more exact treatment of this definition, see (Berberian), but an exposition of this definition is beyond the scope of this paper.

Although the most concise proof of this statement that I am aware of is in (Partington), it relies on the notions of *positive definiteness* and the *polar decomposition* extrapolated directly from the notions of (Axler), Chapter 7, which may be easier to follow.

4.3 Hardy Spaces

Definition 4.3.1 Unit disc (Pohl)**:**

We define the complex unit disc (unit circle) to be $\{ z \in \mathbb{C} : |z| = 1\}$ and denote it by $\mathbb{D}$

Definition 4.3.2 Analytic Function (Cartan)**:**

We define a function f to be *analytic* on an open set if the function has a power series expansion at each point on the set.

■

We would now like to be able to define Hardy Spaces, which we will denote by $\mathbb{H}_p$. At the same time, we don't want to get bogged down in the technicalities of Hardy spaces which would only distract from our discussion of Hankel Operators, to come. Our first simplification is that we will only consider spaces $1 < p < \infty$. Although interesting, $\mathbb{H}_\infty$ is beyond the scope of this paper.

Definition 4.3.3 Hardy Space (Partington)

The Hard space $\mathbb{H}_p$ is the space of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathbb{H}_p} = \sup_{r<1}\left(\int_0^\infty \left|f\left(re^{i\theta}\right)\right| \frac{d\theta}{2\pi}\right)^{\frac{1}{p}} < \infty$$

■

Verbose discussion of this space can be found in (Partington), (Rosenthal), (Varadhan), etc. Here, we will try and avoid that definition as much as possible. Rather, we will present equivalent characterizations or theorems (derived initially from Definition 1.3.3) that are relevant towards our discussion of Hankel Operators.

Lemma 4.3.1: Riesz Factorization Theorem (Partington)[23]

Functions in $\mathbb{H}_1$can be factored into functions in $\mathbb{H}_2$. In particular, $f(z) \in \mathbb{H}_1$ if and only if there exist $g(z), h(z) \in \mathbb{H}_2$ such that $f = gh$ and $\|f\|_{\mathbb{H}_1} = \|g\|_{\mathbb{H}_2}\|h\|_{\mathbb{H}_2}$

■

One of the most import statements we will make before reaching Hankel operators is that there is an isomorphism between Hankel and Hardy spaces. In particular, the following statements follow directly from the Riesz Factorization Theorem.

Lemma 4.3.2: Hankel & Hardy (Partington)

[23] This theorem is easy to state, but its proof in (Partington), while the most remedial I could find, requires Blascke products, which I am afraid to introduce as they are very tangential here

1. $\mathbb{H}_1$ imbeds linearly and isometrically onto $L_1(\mathrm{T})$
2. There is an isomorphism between ℓ_2^+ and $\mathbb{H}_2$,and between ℓ_2^+ and $\mathbb{H}_2^\perp$

■

This is all we will require of Hardy Spaces, though we might note for interest that (Partington) concludes his introduction to the topic with the "Möbius Map." While the content of this theorem is not particularly interesting to us, it is amusing to follow the trail of namesake breadcrumbs consisting of Hankel's professors when building up to the central theorems about Hankel operators.

4.4. Hankel Operators

In this section, we discuss Hankel operators, and describe how they can be explicitly characterized in both $\mathcal{H}$ (Hilbert spaces) and $\mathbb{H}$ (Hardy space).

Definition 4.4.1 Hankel Matrix

For a sequence $s = s_0, s_1, \ldots$, the Hankel matrix $H(s)$ is defined to be the matrix

$$\begin{bmatrix} s_0 & s_1 & s_2 & \cdots \\ s_1 & s_2 & s_3 & \cdots \\ s_2 & s_3 & s_4 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

And $H_n(S)$ is defined to be the upper-left $n \times n$ submatrix of $H(n)$. I will often denote the above matrix $H(\{s_i\})$.

■

Our first task is to characterize Hankel operators on a Hardy space. Where z is the typical standard orthonormal basis, we're looking for a map that takes in z^m and yields $a_m + a_{m+1}z + a_{m+1}z^2$. We denote this mysterious map by Γ following the convention of (Partington)

Theorem 4.4.1 Characterization of the Hankel Matrix over $\mathbb{H}_2$ (Partington)

Let

$$g(z) = \sum_{-\infty}^{\infty} g_k z^k$$

Be in $L_2(\mathrm{T})$, where T is the disc $\{x \,:\, |x| = 1\}$. The function $\Gamma = PM_gR\colon \mathbb{H}_2 \to \mathbb{H}_2$ is a Hankel operator given by $H(\{g_i\})$, where

$$R\left(\sum_0^{\infty} a_n z^n\right) = \sum_0^{\infty} a_n z^{-n} \quad (The \text{ Reversion } map) \;: H_2 \to L_2$$
$$M_g h = gh \;: L_2 \to L_2$$
$$P\left(\sum_{-\infty}^{\infty} c_n z^n\right) = \sum_0^{\infty} c_n z^n \;: L_2 \to H_2$$

Proof (sketch):

Let $R\colon z^m \to z^n$be the "reversion map" in $L_2(\mathrm{T})$ where $\mathrm{T} = \{|x| = 1\}$. Take a function

$$g(z) = \sum_{-\infty}^{\infty} g_k z^k$$

in $L_2(\mathrm{T})$; let $M_g$denote the bounded operator on L_2 which consists of multiplication by g, i.e., let

$$M_g R(z^m) = \sum_{-\infty}^{\infty} g_k z^{-z+k}$$
$$= \sum_{-\infty}^{\infty} (g_{m+\ell^2})^{\ell}$$

Also let P denote the orthogonal projection from $L_2(\mathrm{T})$ onto $\mathbb{H}_2$, which takes negative powers of z to zero. We have that

$$\sum_{-\infty}^{\infty} (g_{m+\ell^2})^{\ell} \to \sum_{0}^{\infty} (g_{m+\ell^2})^{\ell}$$

Thus, Γ; $PM_gR(f)$ will be given by the Hankel matrix $H(a)$ when $g(z) = \sum_{-\infty}^{\infty} g_k z^k$ and $g_k = a_k$ for $k \geq 0$, since $\|P\| = 1$, $\|M_g\| \leq \|g\|_{L_2}$ and $\|R\| = 1$.

■

Theorem 4.4.1 is powerful, because it lets us represent an operator as a Matrix. On the other hand, Nehari's Theorem, outlined below, allows us to map from a Matrix to an operator, which can be equally powerful in some circumstances.

Theorem 4.4.2 Matrix to Operator – Nehari's Theorem (Partington)

If $\Gamma: \mathbb{H}_2 \to \mathbb{H}_2$ is a bounded Hankel operator given by a Hankel matrix $H(\{a_i\})$ with respect to the standard basis, then there exists a function $g \in L_\infty(\mathrm{T})$ such that $\Gamma = PM_gR$ and $\|g\|_{L_\infty}$.

Proof:

We start by looking at inner products on L_2;

$$\langle \Gamma z^n, z^m \rangle = a_{m+n}$$
$$= \langle \Gamma z^{n+m}, 1 \rangle$$

So

$$\left\langle \Gamma \sum_{0}^{N} b_n z^n, \sum_{0}^{M} \bar{c}_m z^m \right\rangle = \left\langle \Gamma \left(\sum_{0}^{N} b_n z^n \right) \left(\sum_{0}^{M} c_m z^m \right), 1 \right\rangle$$

If f_1 and f_2 are polynomials, the linear function $\alpha(f_1, f_2) = \langle \Gamma(f_1, f_2), 1 \rangle = \langle \Gamma f_1, f_2^* \rangle$ satisfies $|\alpha(f_1, f_2)| \leq \|f_1\|_2 \, \|f_2\|_2$.

This allows us to use the "Hann-Banach Theorem", which is used here without proof but can be found, for example, in (Partington). In short, if we notice that polynomials are "dense" in $\mathbb{H}_1$, then we can regard $\mathbb{H}_1$ as a subspace of L_1 and extend α to a linear map $\tilde{\alpha}: L_1 \to \mathbb{C}$ where $\|\tilde{\alpha}\| = \|\alpha\|$.

The Hann-Banach Theorem provides that

$$\tilde{\alpha}(f) = \int_0^{2\pi} f(e^{i\theta})h(e^{i\theta})\frac{d\theta}{2\pi}$$

For some $h \in L_\infty(\mathrm{T})$ where $\|h\|_{L_\infty} = \|\tilde{\alpha}\| \le \|\Gamma\|$. Thus,

$$\begin{aligned} a_{n+m} &= \langle \Gamma z^{m+n}, 1\rangle \\ &= \int_0^{2\pi} e^{i(m+n)\theta}\, h(e^{i\theta})\frac{2\theta}{2\pi} \end{aligned}$$

This means that $h_{-k} = a_k$ when $k \in \{0,1,2,\ldots\}$. If we let $g(e^{i\theta}) = h(e^{-i\theta})$ then $\|g\|_{L_\infty} \le \|\Gamma\|$ and $a_k = g_k$ for $k \in \{0,1,2,\ldots\}$. This completes the proof.

■

Definition 4.4.2 Symbol (Partington)

A function $g \in L_\infty(\mathrm{T})$ such that $g_k = a_k$ for all $k \in \{0,1,2,\ldots\}$ is called a *symbol* for the Hankel operator corresponding to the matrix $H(\{a_i\})$.

Example 4.4.1 Symbol (Partington)

Theorem 1.3.3 tells us that

$$\|\Gamma\| = \left\{\|g\|_{L_\infty} : g \in L_\infty, \text{g a symbol for } \Gamma\right\}$$

Definition 4.4.4 Nehari Extension Problem

The problem of finding a symbol g is called the *Nehari Extension Problem*.

■

Nehari's extension problem is the source of several fascinating, and a few open questions in Operator theory. For example, one can apply Nehari's theorem to show that $H\left(\left\{\frac{1}{n} : n > 0\right\}\right)$ has a norm of π, unexpectedly. This proof is called Hilbert's inequality and was proven by Schur.

Several problems have become famous with regard to Nehari extension; The Caratheodory-Fejer problem asks,

Given a polynomial $a_0 + a_1 z + \cdots a_n z^n$, how do we choose coefficients $a_{n+1} a_{n+1}, \ldots$ to minimize $\left\|\sum_0^\infty a_k z^k\right\|$?

The Nevanlinna-Pick problem asks

Given $z_1, \ldots, z_n$, and $w_1, \ldots, w_n$ complex numbers with $|z_k| < 1$ for all k, how do we find a function such that $f(z_k) = w_k$ for all k that minimizes $\|f\|$?

Interestingly, both of these problems are exactly equivalent to – they reduce to – the Nehari Extension Problem (Partington).

There are plenty of solvable problems introduced by the Nehari Extension problem too; one can find the minimum norm interpolant of f such that $f(0) = 1$ and $f\left(\frac{1}{2}\right) = 0$, as in (Partington).

Lemma 4.4.1 (Partington)

If $f \in \mathbb{H}_2$ and $f \neq 0$, then

$$\int_0^{2\pi} \log |f(e^{i\theta})| \frac{d\theta}{2\pi} > -\infty$$

■

Although the proof of this lemma is very involved, it leads us immediately to two interesting conclusions.

Lemma 4.4.2: Sarason's Lemma: Uniqueness of g (Partington)

If $\Gamma: \mathbb{H}_2 \to \mathbb{H}_2$ corresponds to the Hankel matrix $H(\{a_i\})$, and $(f \in \mathbb{H}_2) \neq 0$ is such that $\|\Gamma f\| = \|\Gamma\| \|f\|$, then

$$g(z) = \frac{(\Gamma f)(z)}{f\left(\frac{1}{z}\right)}$$

Is the unique minimal-norm symbol of Γ.

■

This next theorem also follows from Lemma 1.3.1, and is interesting (1) because of its content, and (2) because of its namesake, Hankel's last professor with Weirstrass at Berlin.

Lemma 4.4.3: Kronecker's Theorem (Partington)[24]

The Hankel matrix $H(\{a_i\})$ has finite rank if and only if $f(z) = a_0 + a_1 z + \cdots$ is a rational function of z

■

We have now characterized Hankel Operators on $\mathbb{H}$. We could go further to characterize compact operators on $\mathbb{H}$ also; we will simply state the main conclusion of that study, which is called Hartman's theorem, and states that $\Gamma = \Gamma_g$ is compact if and only if $g \in \mathbb{H}_\infty^\perp + C(\mathrm{T})$.

By characterizing Hankel operators on $\mathbb{H}$, we have also characterized Hankel operators on $\mathcal{H}$ by way of the isomorphisms in Lemma 1.3.2.

[24] What I've called "Kronecker's Theorem" is really the first part of Kronecker's theorem. The rest of it isn't as interesting, or as accessible.

4.5 Applications of Hankel Operators

Here, we briefly summarize some of the applications of Hankel operators, which are numerous. (Partington) dedicates a quarter of his book to discussing linear systems in terms of Hankel operators, practically. This application repeatedly comes up in real settings. Hankel operators are also used in solving spectral problems, doing control theory, completing matrix contractions, finding singular values, approximating and approximating the inverse of analytic functions, working in Fock and Gaussian spaces, and approximating matrix functions (Peller) (Zhu).

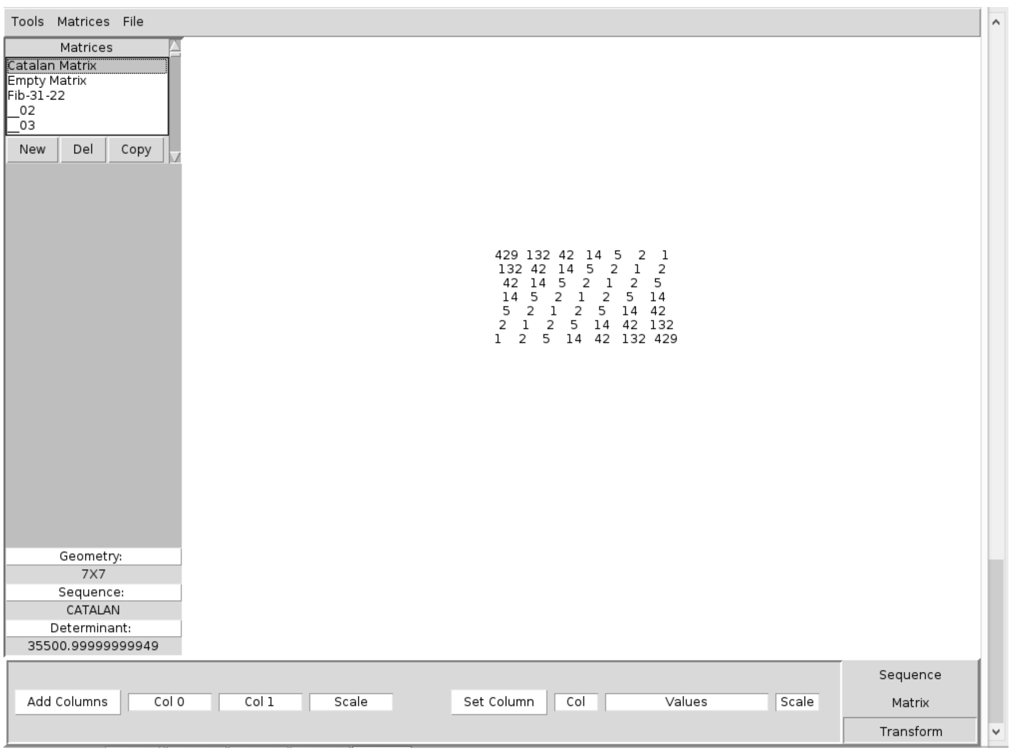
Note that the applications for the Hankel transform are far fewer, and only arise occasionally when trying to invert a Hankel matrix – even then, there are usually more efficient ways to perform the actual inversion.

5. Tools for Proofs

I developed several tools to assist in the proofs and derivations in this paper. Most are quite simple; I am happy to supply the source code or executable program to verify any algorithm or equation presented in Section 3.

One helpful development tool was a matrix editing environment that allowed me to manipulate Matrices and view their qualities. Some of the most useful were, for example, the one that did row/column operations quickly, generated cofactors automatically, or spawned the matrices without having to input values by hand.

Figure 3 Example of interface that aided in derivations. Tools visible include the cofactors automatically generated from the Central Hankel of the Catalan numbers, display of determinant, IO support (naming and saving/loading of matrices), tools to commit row and column operations, tools to enlarge/shrink the matrix, tools to switch sequences, define new sequences, and execute transforms on the sequences, etc.



6. Open Questions

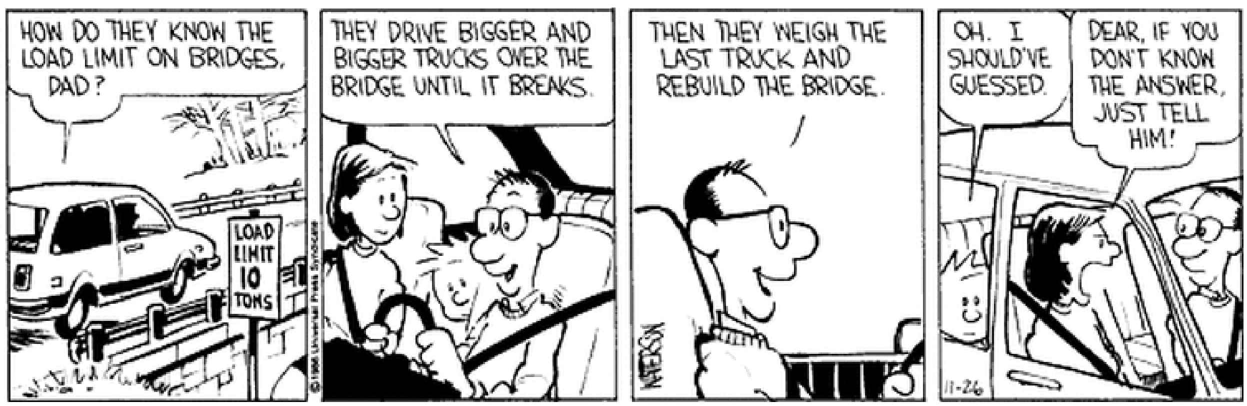
There are several open questions posed by our work. I summarize some of them here.

1. We speculate that $H(L)$ is ∞-injective, i.e., there are infinite matrices that generate the same determinant as $H(L)$. In general, can we describe exactly the injectivity of other sequences?
2. Dougherty was able to compute the Hankel transform of up to 4-sums of Catalan numbers; what happens after that?

3. Cavertivic found the Fibonacci numbers embedded in the Hankel transform of the Catalan numbers; can we find the Catalan numbers embedded in the Toeplitz of the Fibonacci's? Can we expressly describe the Toeplitz transform of the Catalan numbers (I doubt it, but have no proof to show – looking carefully at the transform I saw no pattern)?
4. There appears to be an underlying structure to Toeplitz matrices generated by generalized-Fibonacci sequences. Can it be characterized? Is it more useful to look at the operator-side of this relationship?

7. Concluding Remarks

Thank you to Professor Ben Mathes for guiding me through this process and helping me learn so much about such a fascinating topic.



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