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Quantization of Analysis

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Abstract

In quantum mechanics the replacement of complex vectors with operators is essential to "quantizing" space. Nonetheless, in many physics textbooks there is no justification for this action. Therefore in this thesis I will attempt to understand the mathematical formalism that allows for such a "replacement" to be rigourous. I will approach this topic by first defining a vector spaces and its dual space, a Hilbert space and a conjugate Hilbert space, and an operator space. Next, I will look at the algebraic tensor product of two vector spaces, two Hilbert spaces, and finally two operator spaces. Ultimately we will look at the completion of the tensor product with resect to the minimal norm and show that the minimal norm of a tensor has an analogous inequality to the Cauchy-Schwarz inequality.

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Introduction

In the early Twentieth century, quantum mechanics became the main focus for physicists around the world. This theory helped explain many curious phenomena that could not be described classically. However, the mathematics of this theory was not fully understood until much later. Von Neumann attempted to axiomatize quantum mechanics and realized that it could be considered as a sort of Hilbert space. Furthermore, it was noticed that physical quantities could be represented as linear operators acting on the Hilbert space. Nonetheless, the biggest issue was understanding the replacement of scalar vectors with operators. Ruan's Theorem provided the mathematical structure and justification for physicists to "raise" scalar vectors to operators and from there the theory of operator spaces was developed. However, Ruan's Work is by no means the first "mathematical quantization" but it is asn example of it.

Central to the field of operators space is the notion of a tensor product. The main utility of the tensor product is to create a new vector space out of two existing vector spaces. In this thesis I will study the formulation and the theory behind tensor products and then prove an analogous form of the Cauchy-Schwarz inequality for operators. First, I will look at properties of a vector space and create a new space called the dual space. Then, I will define the universal property of tensor products and prove the existence and uniqueness of the tensor product. Next, I will look at the tensor product of Hilbert spaces and operator spaces. Finally, I will complete the operator space with respect to the minimal norm and then prove the inequality.

1 Vector and Dual Spaces

In order to understand the concept of tensors and tensor products we must first have a firm understanding of vector spaces and dual spaces. In the follow subsections we will define both the vector and dual space and provide proofs of a few necessary and relevant properties of the two spaces. My methods of describing vector spaces closely follows Sheldon Axler's textbook *Linear Algebra Done Right* and my approach to duals spaces parallels Foster and Nightingale's *A short Course in General Relativity* and Nicholas Young's textbook *An Introduction to Hilbert space*. My contribution to this section is filling in the unmentioned steps between lines in the proofs below. [1][2][3]

1.1 Dual Spaces

To begin our understanding of a dual space, I will first give a review on what a vector space is.

Let us denote $\mathbb{F}$ to be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1. *A* ***vector space*** *is a set V along with an addition on V and a scalar* *multiplication on V such that the following properties hold: commutavity*

$$u + v = v + u, \forall u, v \in V \tag{1.1.1}$$

associativity

$$\begin{aligned} (u+v)+w &= u+(v+w), \\ (ab)v &= a(bv) \\ \forall u, v, w &\in V,\ a, b \in \mathbb{F} \end{aligned} \tag{1.1.2}$$

additive identity

$$\textit{there exists an element } 0 \in V \textit{ such that } v + 0 = v, \quad \forall v \in V \tag{1.1.3}$$

additive inverse

$$\textit{for every } v \in V, \textit{ there exists } w \in V \textit{ such that } v + w = 0 \tag{1.1.4}$$

multiplicative identity

$$1v = v, \quad \forall v \in V \tag{1.1.5}$$

distributive properties

$$a(u+v) = au + av, \quad (a+b)u = au + bu, \quad \forall a, b \in \mathbb{F},\ \forall u, v \in V \tag{1.1.6}$$

An example of an n-dimensional vector space, or complex vector space would be $\mathbb{C}^n$. Another example would be the set of polynomials with coefficients in $\mathbb{R}$ or $\mathbb{C}$ given these algebraic properties.

From a vector space, V, we can create a new vector space called the dual space of V. The importance of the dual space is that to define the tensor product we are reliant upon linear and bilinear functionals. To have a concrete understanding of the dual space we will look at the simple case of real-valued functions defined on a real vector space V, $f : V \to \mathbb{R}$. The set of all such functions if given the following algebraic structure

$$(f+g)(\mathbf{v}) = f(\mathbf{v}) + g(\mathbf{v}), \quad \forall \mathbf{v} \in V \tag{1.1.7a}$$

$$(\alpha f)(\mathbf{v}) = \alpha(f(\mathbf{v})) \quad \forall \mathbf{v} \in V \tag{1.1.7b}$$

$$0(\mathbf{v}) = 0 \quad \forall \mathbf{v} \in V \tag{1.1.7c}$$

$$(-f)(\mathbf{v}) = -(f(\mathbf{v})) \quad \forall \mathbf{v} \in V, \tag{1.1.7d}$$

will then satisfy the definition of a vector space given above. Note, that in fact, the set of homomorphisms between two vector spaces is also a vector space. Furthermore, we will restrict the space of all real-valued functions to only those that are linear

$$f(\alpha\mathbf{u}+\beta\mathbf{v})=\alpha f(\mathbf{u})+\beta f(\mathbf{v}) \quad \forall\alpha,\beta\in\mathbb{R},\forall\mathbf{u},\mathbf{v}\in V. \tag{1.1.8}$$

The real-valued linear functions on a real vector space are called linear functionals and if they are given the aforementioned algebraic properties it constitutes a vector space called the dual of V, denoted by V^*.

For simplicity we will point out the two types of vectors, those in V an those in V^*, and distinguish them from one another. For the basis vectors of V^* we will use superscripts and for the components of vectors we will use subscripts, therefore if $\{e^a\}$ is a basis of V^*, then a vector $\lambda\in V^*$ has the unique expression $\lambda=\sum_a\lambda_a e^a$.

In a natural way, we will now prove that the that the dual space of V has the same dimensionality as V.

Lemma 1.1. *The dual space V^* of V has the same dimension as V.*

Proof. First, let V be a N-dimenionsal vector space and let $\{e_a\}$ be the basis of V. We define $\{e^a\}$ to be real-valued functions which maps vectors $\lambda\in V$ into the real number λ^a, which is the ath component relative to the basis vectors of V, $\{e_a\}$. They are given to be

$$e^a(\lambda)=\lambda^a \quad \lambda\in V \tag{1.1.9a}$$
$$e^a(e_b)=\delta^a_b. \tag{1.1.9b}$$

On the last line the term denoted δ^a_b is the Kronecker delta function and is equal to one when $a=b$ and zero otherwise.This clearly gives N real-valued functions which satisfy Eq(1.1.9b). Now we will show that they are linear and they constitute a basis for V^*.

The linearity of the real-valued functions is shown by the restriction

$$e^a(\alpha\lambda+\beta\mu)=\alpha\lambda^a+\beta\mu^a \quad \forall\alpha,\beta\in\mathbb{R} \quad \forall\lambda,\mu\in V \tag{1.1.10}$$

To prove that $\{e^a\}$ is a basis for any given $\nu\in V^*$ we can define N real numbers, denoted ν_a by $\nu(e_a)=\nu_a$. Then

$$\begin{aligned}\nu(\lambda)=\nu(\lambda^a e_a)&=\lambda^a\nu(e_a)\\&=\lambda^a\nu_a\\&=\nu_a e^a(\lambda) \quad \forall\lambda\in V\end{aligned} \tag{1.1.11}$$

Therefore for any $\nu\in V^*$ we have $\nu=\sum\nu_a e^a$, for some ν_a in $\mathbb{R}$, showing that $\{e^a\}$ is the basis of the dual space V^*. The independence of the basis vectors comes from Eq(**??**). Thus $\dim(V^*)=\dim(V)$.

□

Although we have only considered real-valued linear functionals on a real vector space V the proof of the dual and the vector space having the same dimensions can be generalized to other finite dimensional vector spaces. In the infinite dimensional case the dual space has a strictly larger dimension than the original space.

Before we continue, we will first need to define a few terms as they are essential to the following theorems we prove.

Definition 2. *A **norm** on a vector space V is a mapping $\|\cdot\| : V \to \mathbb{R}$ which satisfies the following conditions:*
(i)

$$\|x\| > \quad x \neq 0; \tag{1.1.12}$$

(ii)

$$\|\lambda x\| = |\lambda|\|x\| \quad \forall \lambda \in \mathbb{F},\ x \in V; \tag{1.1.13}$$

(iii)

$$\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in V. \tag{1.1.14}$$

Note that the last condition is precisely the triangle inequality.

Definition 3. *A **metric** is a function that defines the distance between each pair of elements in the set.*

Definition 4. *An **inner product space** is a complex vector space V with an additional structure that is a mapping*

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \tag{1.1.15}$$

such that, for all $x, y, z \in V$ and all $\lambda \in \mathbb{F}$,
(i)

$$\langle x, y \rangle = \langle y, x \rangle^{-}, \tag{1.1.16}$$

where $\langle \cdot, \cdot \rangle^{-}$ is the conjugate of the inner product;
(ii)

$$\langle \lambda x, y \rangle = \lambda \langle x, y \rangle; \tag{1.1.17}$$

(Iii)

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle; \tag{1.1.18}$$

(iv)

$$\langle x, x \rangle > 0 \quad \text{when } x \neq 0. \tag{1.1.19}$$

Definition 5. *A **Hilbert space** $\mathcal{H}$ is an inner product space which is a complete metric space with respect to the metric induced by its inner product.*

Definition 6. *A **normed space** is a pair $(V, \|\cdot\|)$ where V is a real or complex vector space and $\|\cdot\|$ is a norm on V.*

Definition 7. *A **Banach space** is a normed space which is a complete metric space with respect to the metric induced by its norm.*

Theorem 1.2. *Let F be a linear functional on a normed space $(E, \|\cdot\|)$. The following statements are then equivalent to each other:*

1. *F is continuous;*
2. *F is continuous at 0;*
3. *sup $\{|F(x)| : x \in E, \|x\| \leq 1\} < \infty$*

Proof. From statement 1 the proof of statement two is trivial.

Assuming statement 2, given for all $\epsilon > 0$ there is a $\delta > 0$ such that $\|x\| < \delta \Rightarrow |F(x)| < \epsilon$. Take $\epsilon = 1$ then there exists a $\delta > 0$ such that $\|x\| < \delta \Rightarrow |F(x)| < 1$. Therefore $\forall x \in E$ such that $\|x\| \leq 1$ we have $\|\frac{\delta x}{2}\| < \delta \Rightarrow |F(\frac{\delta x}{2})| < 1$. Therefore $|F(x)| < \frac{2}{\delta}$ becauuse of linearity.

Suppose statement 3 and let M be the finite supremum. Taking any pair of $x, y \in E$, $\frac{(x-y)}{\|x-y\|}$ is a unit vector and therefore

$$\begin{aligned} F\left(\frac{(x-y)}{\|x-y\|}\right) &\leq M \\ |F(x-y)| &\leq M\|x-y\| \\ |F(x) - F(y)| &\leq M\|x-y\|; \end{aligned} \tag{1.1.20}$$

the last step follows from linearity of the linear functional. Let $\delta \leq \frac{\epsilon}{M}$ then it is clear that F is continuous.

□

Lemma 1.3. *Let E^* be the set of all continuous linear functionals on the normed space $(E, \|\cdot\|)$ and $F \in E^*$. Then $\|F\| = \sup_{x \in E, \|x\| \leq 1} |F(x)|$ is a norm on E^*.*

Proof. We simply need to check if the norm satisfies the following three conditions:

1. $\|F\| > 0$ if $F \neq 0$;
2. $\|\lambda F\| = |\lambda|\|F\| \quad \forall \lambda \in \mathbb{C}\ \forall F \in E^*$
3. $\|F + F'\| \leq \|F\| + \|F'\| \quad \forall F, F' \in E^*$

Clearly conditions 1 and 2 satisfy the definition of the norm. For condition 3 we have

$$\|F + F'\| = \sup_{x \in E, \|x\| \leq 1} |F(x) + F'(x)|. \tag{1.1.21}$$

By triangle inequality we have

$$\begin{aligned}\sup_{x\in E,\|x\|\le 1}|F(x)+F'(x)| &\le \sup_{x\in E,\|x\|\le 1}\big[|F(x)|+|F'(x)|\big]\\ &\le \sup_{x\in E,\|x\|\le 1}|F(x)|+\sup_{x\in E,\|x\|\le 1}|F'(x)|\\ &= \|F\|+\|F'(x)\| \qquad (1.1.22)\end{aligned}$$

□

Theorem 1.4. *The set E^* of all continuous linear functionals on the normed space $(E, \|\cdot\|)$ is itself a Banach space with respect to pointwise algebraic operations and norm*

$$\|F\| = sup_{x\in E,\|x\|\le 1}\big|F(x)\big| \qquad (1.1.23)$$

Proof. E^* is a vector space over the same field as E. From Theorem 2.3, we know that $\|F\|$ is a real number. Lemma 2.3 also tells us that the norm from Lemma 2.3 is a norm on E^*. We will now show that E is complete.

Let (F_n) be a Cauchy sequence in E^* so that $\|F_n - F_m\| \to 0$ as $n, m \to \infty$. Therefore for all $x \in E \Rightarrow |F_n(x) - |F_m| \to 0$ as n, m go to infinity. We see that this is Cauchy sequence of scalars; we denote the limit of this scalar sequence by $F(x)$. We must now show that $F \in E^*$ and that $F_n \to F$ with respect to the norm on E^*.

Let $\epsilon > 0$ and pick $n_0 \in \mathbb{N}$ such that for all $m, n \ge n_0 \Rightarrow \|F_m - F_n\| < \epsilon$

Then for all $x \in E$ such that $\|x\| \le 1$ and for all $m, n \ge n_0$,

$$|F_m(x) - F_n(x)| < \epsilon \qquad (1.1.24)$$

Letting $m \to \infty, \forall n \ge n_0$ and for all x such that $\|x\| \le 1$,

$$|F(x) - F_n(x)| \le \epsilon \qquad (1.1.25)$$

Since for all $x \in E$ such that $\|x\| \le 1$, $|F(x) - F_n(x)| \le \epsilon$, thereby implies $F - F_0$ and F_{n_0} are bounded, then F is also bounded on the unit ball and by Theorem 2.3 it is continuous. Thus $F \in E^*$. Whenever $n \ge n_0 \Rightarrow \|F - F_n\| \le \epsilon$ implies that $F_n \to F \in E^*$ and so E^* is complete.

□

Theorem 1.5. *For any vector x in a normed space E and any continuous linear functional F on E,*

$$|F(x)| \le \|F\|\|x\|. \qquad (1.1.26)$$

Proof. Suppose that $x \in E$ such that $x \neq 0$. Then $\frac{x}{\|x\|}$ is a unit vector and therefore

$$\begin{aligned} F\left(\frac{x}{\|x\|}\right) &\leq \|F\| \\ \frac{|F(x)|}{\|x\|} &\leq \|F\| \end{aligned} \tag{1.1.27}$$

□

Lemma 1.6. *Let V be an inner product space with an inner product $\langle \cdot, \cdot \rangle$. For all $x, y, z \in V$, if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in V$ then $x = y$.*

Proof. If $\langle x, z \rangle = \langle y, z \rangle$ then

$$\begin{aligned} 0 &= \langle x, z \rangle + (-1)\langle y, z \rangle \\ &= \langle x, z \rangle + \langle -y, z \rangle \\ &\langle x - y, z \rangle \end{aligned} \tag{1.1.28}$$

If this is true for all $z \in V$ then $z = x - y \Rightarrow x - y = 0$.

□

Lemma 1.7. *Let M be a linear subspace of a Hilbert space $\mathcal{H}$ and let $x \in \mathcal{H}$. Then $x \in M^{\perp}$ if and only if*

$$\|x - y\| \geq \|x\| \quad \forall y \in M \tag{1.1.29}$$

Proof. ($\Rightarrow$) If $x \in M^{\perp}$ then, $\forall y \in M$ x and y are orthogonal then by the Pythagoras' theorem we have

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 \leq \|x\|^2 \tag{1.1.30}$$

($\Leftarrow$) Suppose that Eq(1.1.29) holds. for all $y \in M$ and for all $\lambda \in \mathbb{F}$, then $\lambda y \in M$ and

$$\|x - \lambda y\|^2 \geq \|x\|^2. \tag{1.1.31}$$

Rewriting this as an inner product we have

$$\begin{aligned} \langle x - \lambda y, x - \lambda y \rangle &\geq \|x\|^2 \\ \|x\|^2 - \bar{\lambda}\langle x, y \rangle - \lambda \langle x, y \rangle^{-} + |\lambda|^2 \|y\|^2 &\geq \|x\|^2 \\ -2\mathrm{Re}\bar{\lambda}\langle x, y \rangle + |\lambda|^2 \|y\|^2 &\geq 0. \end{aligned} \tag{1.1.32}$$

The last line comes from $\forall z \in \mathbb{C} \Rightarrow z + \bar{z} = 2\mathrm{Re}\{z\}$. Eq(1.1.32) holds for any $\lambda \in \mathbb{C}$, it also holds for $\lambda = tz$ where $t > 0$ and $z \in \mathbb{C}$, where $|z| = 1$, then $\bar{z}(x, y) = |(x, y)|$. Thus

$$\begin{aligned} -2t|\langle x, y\rangle| + t^2\|y\|^2 &\geq 0 \\ \|\langle x, y\rangle\| &\leq \frac{1}{2}t\|y\|^2. \end{aligned} \tag{1.1.33}$$

Letting $t \to 0$ the result shows $(x, y) = 0$.

□

Lemma 1.8. *Let M be a closed linear subspace of a Hilbert space H and let $x \in H$. There exists $y \in M$ $z \in M^{\perp}$ such that $x = y + z$.*

Proof. Take $y \in M$ and let it be the closest point to x in M, by definition

$$\|x - y\| \leq \|x - m\| \quad \forall m \in M. \tag{1.1.34}$$

Let $z = x - y$ then $x = y + z$. $\forall m \in M \Rightarrow y + m \in M$ and so

$$\begin{aligned} \|z\| = \|x - y\| &\leq \|x - (y + m)\| \\ &\leq \|z - m\| \quad \forall m \in M. \end{aligned} \tag{1.1.35}$$

By Lemma 2.7, $z \in M^{\perp}$.

□

Theorem 1.9. *Riesz-Frechet Theorem*

Let H be a Hilbert space and let F be a continuous linear functional on H. There exists a unique $y \in H$ such that

$$F(x) = (x, y) \quad \forall x \in H. \tag{1.1.36}$$

Furthermore $\|y\| = \|F\|$.

Proof. From Lemma 2.6 we know that y is unique because

$$(x, y) = F(x) = (x, y') \quad \forall x \in H \tag{1.1.37}$$

implies $y = y'$.

A trivial case of F is when F is the zero operator and we take $y = 0$. Now let us assume the case were the functional F is not equal to the zero operator.

Let M be the kernel of the linear functional F,

$$M = \mathrm{Ker}F = \{x \in H : F(x) = 0\}, \tag{1.1.38}$$

and M is a proper closed subspace of H by the continuity of F. From Lemma 2.8, we know that $H = M \oplus M^{\perp}$ and therefore $M^{\perp} \neq \{0\}$. Let $z \in M^{\perp}$ such that $z \neq 0$. We can scale z with a scalar such that $F(z) = 1$. Then pick a $z \in M^{\perp}$ such that for any $x \in H$

$$x = (x - F(x)z) + F(x)z \tag{1.1.39}$$

Because H is a direct some of M and its orthogonal complement $M^{\perp}$, the first term on the right hand expression is an element of M and the second term is an element of $M^{\perp}$. Taking an inner product of both sides with z,

$$(x, z) = (F(x)z, z) = F(x)\|z\|^2 \quad \forall x \in H, \tag{1.1.40}$$

since $z \perp M$. If we let $y = \frac{z}{\|z\|^2}$ we have

$$(x, y) = F(x) \tag{1.1.41}$$

If $\|x\| \leq 1$ then by Cauchy-Schwarz,

$$|F(x)| = |(x, y)| \leq \|x\|\|y\| \leq \|y\| \tag{1.1.42}$$

Let $x = \frac{y}{\|y\|}$, which is a unit vector, and therefore

$$\|F\| \geq |F(x)| = \frac{|F(y)|}{\|y\|} = \frac{|(y, y)|}{\|y\|} = \|y\|. \tag{1.1.43}$$

Therefore $\|F\| = \|y\|$.

$\square$

2 Tensor Products

Previously, in the sections above, we have defined a vector space and created a new space called its dual. However, in the general case, looking at infinite dimensional vector spaces, we cannot simply repeat this process. Therefore we introduce the tensor product and it generates a new vector space from two other vector spaces. Primarily, tensor products came around for vector spaces due to its inherent need in physics and engineering, nonetheless it is a vital aspect of functional analysis and operator space theory. We will use tensor products in order to formally prove the generalization of the Cauchy-Schwarz inequality of complex vectors to that of operators. The following description of tensor products is of vector spaces, however it can be extended to that of Hilbert spaces and ultimately operator spaces. When building the tensor product space we are actually constructing the algebraic tensor product and therefore the resultant space is not necessarily complete. Professor Ben Mathes and I constructed the following proofs on the tensor product.

2.1 Building the Tensor Product

To begin the construction of tensor products of vectors spaces over a field, I will first define the term linear extension.

Definition 8. *Let $\mathcal{E} \subset V$ be a basis of V. Let $f_0 : \mathcal{E} \to W$ be a function. The* ***linear extension*** *f of f_0 to V is defined by*

$$f\Big(\sum_{e\in\mathcal{E}} \alpha_e e\Big) \equiv \sum_{e\in\mathcal{E}} \alpha_e f_0(e), \tag{2.1.1}$$

where $\alpha_e \in \mathbb{F}$ and α_e is finitely non-zero.

Next let us define bilinear and the universal property of tensor products.

Let V_1, V_2, W be vector spaces over a field $\mathbb{F}$. A map $\phi : V_1 \times V_2 \to W$ is called bilinear if for all $x, \alpha \in V_1, y, \beta \in V_2$, the maps that take

$$x \mapsto \phi(x, \beta), \quad y \mapsto \phi(\alpha, y) \tag{2.1.2}$$

holding α and β constant are linear maps. Linearity of the maps can be easily shown, let the map $f(x) : V_1 \to W$ such that $f(x) = \phi(x, \beta)$ then

$$f(ax) = \phi(ax, \beta) = a\phi(x, \beta) \tag{2.1.3a}$$

$$f(x + y) = \phi(x + y, \beta) = \phi(x, \beta) + \phi(y, \beta) \tag{2.1.3b}$$

where it holds for any $a \in \mathbb{F}, \beta \in V_2, x, y \in V_1$.

A bilinear map from $\phi : V_1 \times V_2 \to W$ has the universal property of tensor products when $\forall$ bilinear maps $\psi : V_1 \times V_2 \to W'$ there is a unique linear map from $\hat{\psi} : W \to W'$

$$\hat{\psi}\big(\phi(x, y)\big) = \psi(x, y), \quad \forall x \in V_1, y \in V_2. \tag{2.1.4}$$

Now, we will prove several properties of tensor products. To prove the theorem on existence we will first prove a Lemma that provides a sufficient condition.

Lemma 2.1. *Let $\mathcal{E}$ and $\mathcal{F}$ be the basis of V_1 and V_2 respectively. If $\phi : V_1 \times V_2 \to W$ has $\{\phi(x, y)|x \in V_1, y \in V_2\}$ as a spanning set of W and if ϕ has the property that $\{\phi(e, f)|e \in \mathcal{E}, f \in \mathcal{F}\}$ is linearly independent whenever $\mathcal{E} \subseteq V_1$ and $\mathcal{F} \subseteq V_2$ are linear independent, then ϕ satisfies the universal product.*

Proof. First assume that $\mathcal{E}$ and $\mathcal{F}$ are the basis of V_1 and V_2 respectively and that $\phi : V_1 \times V_2 \to W$ has $\{\phi(x, y)|x \in V_1, y \in V_2\}$ as a spanning set of W. Now consider the diagram

$$\begin{tikzcd}
V_1 \times V_2 \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & W \\
& W' &
\end{tikzcd}$$

Suppose $\mathcal{E}$ and $\mathcal{F}$ are both linearly independent and that $\{\phi(e,f)|e \in \mathcal{E}, f \in \mathcal{F}\}$ is linearly independent whenever $\mathcal{E} \subseteq V_1$ and $\mathcal{F} \subseteq V_2$ are linear independent.

So $\{\phi(e,f)|e \in \mathcal{E}, f \in \mathcal{F}\}$ is linearly independent and $\phi(x,y) = \phi\big(\sum_{e\in\mathcal{E}} \alpha_e e, \sum_{f\in\mathcal{F}} \beta_f f\big)$ and by bilinearity we have

$$\phi\Big(\sum_{e\in\mathcal{E}} \alpha_e e, \sum_{f\in\mathcal{F}} \beta_f f\Big) = \sum_{e\in\mathcal{E}, f\in\mathcal{F}} \alpha_e \beta_f \phi(e,f) \tag{2.1.5}$$

so $\{\phi(e,f)|e \in \mathcal{E}, f \in \mathcal{F}\}$ spans $\{\phi(x,y)|x \in V_1, y \in V_2\}$ and this spans W therefore $\{\phi(e,f)|e \in \mathcal{E}, f \in \mathcal{F}\}$ is a basis of W. Now to finish the proof note that a bilinear function is defined by where it sends the basis vectors of the vector space. Therefore if we let $\hat{\psi}\big(\phi(e,f)\big) = \psi(e,f)$ and we extend linearly we will have a unique linear function that makes the diagram commute.

□

Now let us combine our knowledge of vector and duals spaces with the notion of tensor products.

Theorem 2.2. *The tensor product exists.*

Proof. Let V_1 and V_2 be vector spaces.

Consider a bilinear mapping

$$\phi : V_1 \times V_2 \to \mathrm{BL}[V_1^* \times V_2^* \to \mathbb{F}] = \{\phi^* : V_1^* \times V_2^* \to \mathbb{F} | \phi^* \text{ is bilinear}\}, \tag{2.1.6}$$

that is defined as such

$$\phi_{(\mu,\nu)}(x,y) = \mu(x)\nu(y). \tag{2.1.7}$$

The bilinear function ϕ is dependent upon the linear functionals μ, ν which are elements of the duals of V_1 and V_2 respectively. It is clear that the map is bilinear

$$\begin{aligned}
\phi_{(\mu,\nu)}(\alpha x + \beta y, z) &= \mu(\alpha x + \beta y)\nu(z) \\
&= \alpha\mu(x)\nu(z) + \beta\mu(y)\nu(z) && (2.1.8a) \\
\phi_{(\mu,\nu)}(x, \alpha y + \beta z) &= \mu(x)\nu(\alpha y + \beta z) \\
&= \alpha\mu(x)\nu(y) + \beta\mu(x)\nu(z), && (2.1.8b)
\end{aligned}$$

for all $x \in V_1, y \in V_2, \alpha, \beta \in \mathbb{F}, \mu \in V_1^*, \nu \in V_2^*$. Let both $\mathcal{E}$ and $\mathcal{F}$ be a linearly independent bases of V_1 andV_2 respectively then from above .

Then to show the linear independence of the space $\mathrm{BL}[V_1^* \times V_2^* \to \mathbb{F}]$ we set an arbitrary linear combination of the basis vectors $\sum_{e \in \mathcal{E}, f \in \mathcal{F}} \alpha_{(e,f)} \phi(e, f)$ in $\mathrm{BL}[V_1^* \times V_2^* \to \mathbb{F}]$ equal to zero and solve for the constant coefficients:

$$\begin{aligned} 0 &= \sum_{e \in \mathcal{E}, f \in \mathcal{F}} \alpha_{(e,f)} \phi(e, f) \\ \Rightarrow &\sum_{e \in \mathcal{E}, f \in \mathcal{F}} \alpha_{(e,f)} \phi(e, f)(e_0^*, f_0^*) \\ = &\sum_{e \in \mathcal{E}, f \in \mathcal{F}} \alpha_{(e,f)} e_0^*(e) f_0^*(f) = 0 \\ &\Rightarrow \alpha_{(e_0, f_0)} = \alpha = 0. \end{aligned} \tag{2.1.9}$$

In this case we have defined ϕ to be dependent upon the basis vectors e_0^* and f_0^* of the dual spaces respectively. Since we have shown that all coefficients of the linear combination are zero this implies that the basis vectors are linearly independent and thus fulfills our condition given by Lemma 3.1 of the existence of the tensor product. □

Theorem 2.3. *If $\phi : V_1 \times V_2 \to W$ and $\phi' : V_1 \times V_2 \to W'$ both have the universal property then W is isomorphic to W' via an isomorphism.*

Proof. Suppose $\phi : V_1 \times V_2 \to W$ and $\phi' : V_1 \times V_2 \to W'$ have the universal property of tensor products. Now consider the diagram

$$\begin{tikzcd} & W \\ V_1 \times V_2 \arrow[ur, "\phi"] \arrow[dr, "\phi'"'] & \\ & W' \end{tikzcd}$$

but because both bilinear functionals ϕ and ϕ' have the universal property of the tensor product our diagram can be rewritten as

$$
\begin{tikzcd}
 & W \arrow[d, "f"] \\
V_2 \times V_2 \arrow[ur, "\phi"] \arrow[r, "\phi'"] \arrow[dr, "\phi"'] & W' \arrow[d, "f'"] \\
 & W
\end{tikzcd}
$$

where f and f' are unique linear functionals. Notice that $f \circ f'$ and $f' \circ f$ are the identity on W. Therefore f and f' are inverses of each other and W W' are isomorphic to each other.

$\square$

Note that because of the uniqueness up to an isomorphism, when you generate the tensor product of two spaces it is the only tensor product of the two spaces.

Now let us introduce the notion of the "conjugate" of a Hilbert space $\mathcal{H}$. For a "normal" Hilbert space, $\mathcal{H}$ has the algebraic structure and inner product defined by the mappings

$$(x, y) \to x + y : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \tag{2.1.10a}$$
$$(a, x) \to ax : \mathbb{C} \times \mathcal{H} \to \mathcal{H} \tag{2.1.10b}$$
$$(x, y) \to \langle x, y \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}. \tag{2.1.10c}$$

However we define the "conjugate" Hilbert space with a slight twist.

Definition 9. *The* ***conjugate Hilbert space*** $\bar{\mathcal{H}}$ *is the same set* $\mathcal{H}$*, with the algebraic structure and inner product defined by the mappings:*

$$(x, y) \to x + y : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \tag{2.1.11a}$$
$$(a, x) \to a \bar{\cdot} x = \bar{a}x : \mathbb{C} \times \mathcal{H} \to \mathcal{H} \tag{2.1.11b}$$
$$(x, y) \to \langle x, y \rangle^{-} = \overline{\langle x, y \rangle} : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \tag{2.1.11c}$$

where

$$\langle x, y \rangle^{-} = \overline{\langle x, y \rangle} = \langle y, x \rangle. \tag{2.1.12}$$

Now we will look at the tensor products of two Hilbert spaces, $\mathcal{H}$ and $\bar{\mathcal{H}}$, $\bar{\mathcal{H}}$ being the conjugate linear Hilbert space. An elementary tensor of $\mathcal{H} \otimes \bar{\mathcal{H}}$ is denoted as $x \otimes y$ and is an element of the bounded linear operators on $\mathcal{H}$. We define the elementary tensor on an element of $\mathcal{H}$ and the inner product of two tensors by

$$x \otimes y(z) = \langle z, y \rangle x; \tag{2.1.13a}$$

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle \tag{2.1.13b}$$

for some $x, z \in \mathcal{H}, y \in \bar{\mathcal{H}}$.

Now we will prove a simple theorem about the tensor product of Hilbert spaces $\mathcal{H} \otimes \bar{\mathcal{H}}$.

Theorem 2.4. *Let $\{e_i\}_{i=1}^n, \{f_i\}_{i=1}^m$ be independent sets of $\mathcal{H}$ and $\bar{\mathcal{H}}$ respectively. The set $\{e_i \otimes f_i\}_{ij}$ is independent.*

Proof. It is sufficient to show that

$$\sum_{ij} \alpha_{ij} e_i \otimes f_j = 0 \tag{2.1.14}$$

such that $\alpha_{ij} = 0$ for all i, j. Given j_0 there exists a $z \in \mathcal{H}$ such that z is perpendicular to the independent set

$$\{f_1, ..., f_{j_0 - 1}, f_{j0+1}, ..., f_m\}. \tag{2.1.15}$$

so that

$$\langle z, f_{j0} \rangle \neq 0. \tag{2.1.16}$$

Now we have

$$\sum_{ij} \alpha_{ij} e_i \otimes f_j(z) = \sum_{i}^{n} \alpha_{ij_0} \langle z, f_0 \rangle e_i \tag{2.1.17}$$

which implies that

$$\alpha_{ij_0} = 0 \tag{2.1.18}$$

for all i because the set $\{e_i\}$ is independent and the choice of j_0 is arbitrary. Using a similar argument for the other half completes the proof.

□

Next we will prove two lemmas that will show the relationship between tensors and operators.

Lemma 2.5. *The conjugate Hilbert space is isomorphic to the dual space of the Hilbert space $\mathcal{H}$. Furthermore, this relationship is linear.*

Proof. Let there be a mapping $\psi : \bar{\mathcal{H}} \to \mathcal{H}^*$. Then we define the mapping to be

$$\psi(y) = \langle \ , y \rangle_{\mathcal{H}}. \tag{2.1.19}$$

Therefore this is the desired isomorphism. Finally, it is linear as

$$\alpha \bar{\cdot} y \mapsto \langle \ , \bar{\alpha} y \rangle = \alpha \langle \ , y \rangle. \tag{2.1.20}$$

□

An intuitive approach to our next lemma can be seen if one were to associate the conjugate Hilbert space with the dual of the Hilbert space.

Lemma 2.6. *The space of finite rank operators are isomorphic to the tensor product* $\mathcal{H} \otimes \bar{\mathcal{H}}$.

Proof. Let V, W be Hilbert spaces and let $B(V, W)$ be the set of all bounded operators from V to W. Then let ϕ be a mapping from $W \otimes \overline{V}$ to $B(V, W)$, denoted $\phi : W \otimes \overline{V} \to B(V, W)$. We define this mapping to be

$$\phi(w, \lambda) : v \mapsto \big(\langle v, \lambda \rangle\big) w. \tag{2.1.21}$$

Thus we have our desired isomorphism.

□

In our next step, we will consider the tensor product of bounded linear operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.

Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{H})$. Then the tensor product of A and B, denoted as $A \otimes B$ is an operator defined as

$$A \otimes B : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}. \tag{2.1.22}$$

We define the operator acting on a tensor as

$$A \otimes B(x \otimes y) \equiv Ax \otimes By, \tag{2.1.23}$$

and by linearity of the operator we have

$$A \otimes B\big(\sum_i^n x_i \otimes y_i\big) = \sum_i^n Ax_i \otimes By_i. \tag{2.1.24}$$

We will now prove that the A operator and B operator on a tensor $x \otimes y$ act as a left multiplier and a conjugate right multiplier respectively for finite rank operators on a Hilbert space:

$$L_A \simeq A \otimes I_{\bar{\mathcal{H}}}, \quad R_B \simeq I_{\mathcal{H}} \otimes B^*. \tag{2.1.25}$$

Theorem 2.7. *Let A be a bounded linear operator on $\mathcal{H}$ and B a bounded linear operator on $\bar{\mathcal{H}}$. A is then the left multiplier and B is the conjugate right multiplier.*

Proof. Let $x \otimes y$ be an element of $\mathcal{H} \otimes \bar{\mathcal{H}}$.

$$\begin{aligned} A(x \otimes y)(z) &= A(\langle z, y\rangle x) \\ &= \langle z, y\rangle A(x) \\ &= A(x) \otimes y(z) \\ &= A \otimes I_{\bar{\mathcal{H}}}(x \otimes y)(z) \\ &= L_A(x \otimes y)(z). \end{aligned} \tag{2.1.26}$$

Now for the conjugate right multiplier B

$$\begin{aligned} (x \otimes y)B(z) &= x \otimes y(Bz) \\ &= \langle B(z), y\rangle x \\ &= \langle z, B^*(y)\rangle x \\ &= x \otimes B^* y(z) \\ &= I_{\mathcal{H}} \otimes B^*(x \otimes y)(z) \\ &= R_B(x \otimes y)(z) \end{aligned} \tag{2.1.27}$$

□

We will now prove a simple theorem for tensors that are isomorphic to the Hilbert-Schmidt operators, however in order to do so we will first prove a lemma.

Lemma 2.8. *Let $x \otimes y \in \mathcal{H} \otimes \bar{\mathcal{H}}$.*

$$Tr(x \otimes y) = \langle x, y\rangle. \tag{2.1.28}$$

Proof. First note that clearly

$$(x \otimes y)_{ij} = x_i y_j, \tag{2.1.29}$$

then applying the trace operation to $x \otimes y$ we obtain

$$\begin{aligned} \mathrm{Tr}(x \otimes y) &= \sum_i (x \otimes y)_{ii} \\ &= \sum_i x_i y_i \\ &= \langle x, y\rangle. \end{aligned} \tag{2.1.30}$$

□

Theorem 2.9. *The tensor product of the Hilbert space $\mathcal{H}$ and the conjugate Hilbert space $\bar{\mathcal{H}}$, denoted as $\mathcal{H} \otimes \bar{\mathcal{H}}$, is isomorphic to the space of Hilbert-Schmidt Operators.*

Proof. It is sufficient to show that

$$\mathrm{Tr}\big((x_2 \otimes y_2)^*(x_1 \otimes y_1)\big) = \langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle, \tag{2.1.31}$$

where $x_1 \otimes y_1, x_2 \otimes y_2 \in \mathcal{H} \otimes \bar{\mathcal{H}}$.

$$\begin{aligned}
\mathrm{Tr}\big((x_2 \otimes y_2)^*(x_1 \otimes y_1)\big) &= \mathrm{Tr}\big((y_2 \otimes x_2)(x_1 \otimes y_1)\big) \\
&= \mathrm{Tr}\big((y_2 \otimes x_2(x_1)) \otimes y_1\big) \\
&= \langle y_2 \otimes x_2(x_1), y_1 \rangle \\
&= \langle \langle x_1, x_2 \rangle y_2, y_1 \rangle \\
&= \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle \\
&= \langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle
\end{aligned} \tag{2.1.32}$$

□

2.2 Tensor products and the Hilbert-Schmidt Class

From a general construction of the algebraic tensor products we will look at specific examples of tensor products limited to Hilbert spaces. I will formally show that the tensor product of Hilbert spaces is a Hilbert space and actually is the space of Hilbert-Schmidt functionals on the conjugate Hilbert space of the cartesian product of the Hilbert spaces. My approach closely follows Kadison and Ringrose's *Fundamentals of the Theory of Operator Algebras*. My work in this section also adding in the steps to each proof.[4]

In order to begin, we first will prove a basic theorem that is essential to understanding Hilbert spaces.

Theorem 2.10. *If Y is an orthonormal set in a Hilbert space $\mathcal{H}$, the following three conditions are equivalent:*

(i)

$$\forall u \in \mathcal{H}, \quad u = \sum_{y \in Y} \langle u, y \rangle y; \tag{2.2.1}$$

(ii)

$$\forall u, v \in \mathcal{H}, \quad \langle u, v \rangle = \sum_{y \in Y} \langle u, y \rangle \langle y, v \rangle; \tag{2.2.2}$$

(iii)

$$\forall u \in \mathcal{H}, \|u\|^2 = \sum_{y \in Y} |\langle u, y\rangle|^2; \tag{2.2.3}$$

Proof. $(i \to ii)$: Let $u, v \in \mathcal{H}$ such that $u = \sum_{y \in Y} \langle u, y\rangle y$ and $v = \sum_{y \in Y} \langle v, y\rangle y$. Then we have

$$\begin{aligned}
\langle u, v\rangle &= \left\langle \sum_{y \in Y} \langle u, y\rangle y, \sum_{y \in Y} \langle v, y\rangle y \right\rangle \\
&= \sum_{y \in Y} \left\langle \langle u, y\rangle y, \langle v, y\rangle y \right\rangle \\
&= \sum_{y \in Y} \langle u, y\rangle \overline{\langle v, y\rangle} \langle y, y\rangle \\
&= \sum_{y \in Y} \langle u, y\rangle \langle y, v\rangle
\end{aligned} \tag{2.2.4}$$

Therefore i) implies ii).

$(ii \to iii)$: Let $u, v \in \mathcal{H}$ such that the inner product is defined to be $\langle u, v\rangle = \sum_{y \in Y} \langle u, y\rangle \langle y, v\rangle$. Now to find the square of the norm we have

$$\begin{aligned}
\|u\|^2 &= \langle u, u\rangle \\
&= \sum_{y \in Y} \langle u, y\rangle \langle y, u\rangle \\
&= \sum_{y \in Y} |\langle u, y\rangle|^2
\end{aligned} \tag{2.2.5}$$

Therefore ii) implies iii).

$(iii \to i)$: Let $u \in \mathcal{H}$ such that $\|u\|^2 = \sum_{y \in Y} |\langle u, y\rangle|^2$. Then to solve for u in terms of the basis vectors, y, of Y we have

$$\begin{aligned}
\|u\|^2 &= \sum_{y \in Y} |\langle u, y\rangle|^2 \\
&= \sum_{y \in Y} \langle u, y\rangle \langle y, u\rangle \\
&= \sum_{y \in Y} \langle u, y\rangle \langle y, u\rangle \langle y, y\rangle \\
&= \sum_{y \in Y} \left\langle \langle u, y\rangle y, \langle u, y\rangle y \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle u, u \rangle \\
&\Rightarrow u = \langle u, y \rangle y \qquad (2.2.6)
\end{aligned}
$$

Therefore iii) implies i) and the proof of the theorem is complete. On a side note if Y is an orthonormal basis, then condition ii) is also known as Parseval's equation.

□

In defining the (Hilbert) tensor product $\mathcal{H}$ of two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, the approach we take will utilize the "universal" property of the tensor product. In terms of operators, rather than mappings, the Hilbert space $\mathcal{H}$ is characterized, up to isomorphism, by the existence of a bilinear mapping $p : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}$. It has the following property that each "suitable" bilinear mapping L from $\mathcal{H}_1 \times \mathcal{H}_2$ into a Hilbert space $\mathcal{K}$ has a unique factorization $L = Tp$, with T being a bounded linear operator from $\mathcal{H}$ into $\mathcal{K}$.

Before the formal construction of the theory, we need to understand the intuitive aspects of it. When $x_1 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2$, we want to view the element $p(x_1, x_2) \in \mathcal{H}$ as a "product" $x_1 \otimes x_2$ of sorts. Later we will see the parallels between the formal product and the tensor product. The linear combinations of such products form an everywhere-dense subspace of $\mathcal{H}$. The bilinearity of p implies that these products satisfy certain linear relations:

$$
\begin{aligned}
(x_1 + y_1) \otimes (x_2 + y_2) - x_1 \otimes x_2 - x_1 \otimes y_2 - y_1 \otimes x_2 - y_1 \otimes y_2 = 0 \\
x_1, y_1 \in \mathcal{H}_1, x_2, y_2 \in \mathcal{H}_2 \qquad (2.27)
\end{aligned}
$$

All the linear relations satisfied by product vectors can be achieved by use of the bilinearity of p. The inner product on $\mathcal{H}$ satisfies the conditions:

$$
\begin{aligned}
\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle &= \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle \qquad (2.2.8a) \\
\|x_1 \otimes x_2\|^2 &= \langle x_1 \otimes x_2, x_1 \otimes x_2 \rangle \\
&= \langle x_1, x_1 \rangle \langle x_2, x_2 \rangle = \|x_1\|^2 \|x_2\|^2 \\
\|x_1 \otimes x_2\| &= \|x_1\| \|x_2\|. \qquad (2.2.8b)
\end{aligned}
$$

Our construction of the Hilbert space $\mathcal{H}$, the elements of $\mathcal{H}$ are complex-valued functions defined on the product $\mathcal{H}_1 \times \mathcal{H}_2$ and conjugate-linear in both variables. If $v_1 \in \mathcal{H}_1, v_2 \in \mathcal{H}_2$, $v_1 \otimes v_2$ is the function that assigns the value $\langle v_1, x_1 \rangle \langle v_2, x_2 \rangle$ to the element $(x_1, x_2) \in \mathcal{H}_1 \times \mathcal{H}_2$.

Now, suppose that $\mathcal{H}_1, ..., \mathcal{H}_n$ are Hilbert spaces and ϕ is a mapping from the cartoon product of all these Hilbert spaces into the scalar field $\mathbb{C}$. The mapping ϕ is called a bounded multilinear functional on $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ if ϕ is linear in each of its variables, assuming that the other variables remain fixed, and there is a real number c such that

$$|\phi(x_1, ..., x_n)| \leq c\|x_1\| \cdots \|x_n\|, \quad x_1 \in \mathcal{H}_1, ..., x_n \in \mathcal{H}_n. \tag{2.2.9}$$

If this is so then the lest such constant c is denoted by the norm of the mapping $\|\phi\|$. Then ϕ is a continuous mapping from $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathbb{C}$ relative to the product tor the norm topologies on the Hilbert spaces.

Theorem 2.11. *Suppose that $\mathcal{H}_1, ..., \mathcal{H}_n$ are Hilbert spaces and ϕ is a bounded multilinear functional on $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$.*

(i) The sum

$$\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\phi(y_1, ..., y_n)|^2 \tag{2.2.10}$$

has the same finite or infinite value for all orthonormal bases Y_1 of $\mathcal{H}_1$,..., Y_n of $\mathcal{H}_n$.

(ii) If $\mathcal{K}_1, ..., \mathcal{K}_n$ are Hilbert spaces, $A_m \in \mathcal{B}(\mathcal{H}_m, \mathcal{K}_m)$, $(m = 1, ..., n)$, ψ is a bounded multilinear functional on $\mathcal{K}_1 \times \cdots \mathcal{K}_n$, and

$$\begin{aligned} \phi(x_1, ..., x_n) = \psi(A_1 x_1, ..., A_n x_n) \\ x_1 \in \mathcal{H}_1, ..., x_n \in \mathcal{H}_n \end{aligned} \tag{2.2.11}$$

then

$$\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\phi(y_1, ..., y_n)|^2 \leq \|A_1\|^2 \cdots \|A_n\|^2 \sum_{z_1 \in Z_1} \cdots \sum_{z_n \in Z_n} |\phi(z_1, ..., z_n)|^2, \tag{2.2.12}$$

when Y_m and Z_m are orthonormal bases of $\mathcal{H}_m$ and $\mathcal{K}_m$, respectively where $m = 1, ..., n$.

Proof. To prove (i) it is sufficient to show that

$$\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\phi(y_1, ..., y_n)|^2 \leq \sum_{z_1 \in Z_1} \cdots \sum_{z_n \in Z_n} |\phi(z_1, ..., z_n)|^2, \tag{2.2.13}$$

whenever Y_m, Z_m are orthonormal bases of $\mathcal{H}_m, m = 1, ..., n$. Note that since Y_m, Z_m are orthonormal bases we can represent the basis vectors in one in terms of a linear combination of the other.

$$y_1 = \sum_{z_1 \in Z_1} \langle y_1, z_1 \rangle z_1, ..., y_n = \sum_{z_n \in Z_n} \langle y_n, z_n \rangle z_n \tag{2.2.14}$$

Therefore,

$$
\begin{aligned}
\sum_{y_1\in Y_1}\cdots\sum_{y_n\in Y_n}|\phi(y_1,...,y_n)|^2 &= \sum_{y_1\in Y_1}\cdots\sum_{y_n\in Y_n}|\phi(\sum_{z_1\in Z_1}\langle y_1,z_1\rangle z_1,...,\sum_{z_n\in Z_n}\langle y_n,z_n\rangle z_n)|^2 \\
&= \sum_{y_1\in Y_1}\cdots\sum_{y_n\in Y_n}|\sum_{z_1\in Z_1}\cdots\sum_{z_n\in Z_n}\langle y_1,z_1\rangle\cdots\langle y_n,z_n\rangle\phi(z_1,...,z_n)|^2 \\
&\le \sum_{z_1\in Z_1}\cdots\sum_{z_n\in Z_n}\|z_1\|^2\cdots\|z_n\|^2|\phi(z_1,...,z_n)|^2 \\
&= \sum_{z_1\in Z_1}\cdots\sum_{z_n\in Z_n}|\phi(z_1,...,z_n)|^2
\end{aligned}
\tag{2.2.15}
$$

The third line is arrived by noticing that we can rewrite the $z_m \in Z_m$ basis in terms of $y_m \in Y_m$

$$
z_1 = \sum_{y_1\in Y_1}\langle z_1,y_1\rangle y_1,...,z_n = \sum_{y_n\in Y_n}\langle z_n,y_n\rangle y_n. \tag{2.2.16}
$$

Also note that by definition $\langle y_m, z_m\rangle = \overline{\langle z_m, y_m\rangle}$ and that

$$
\begin{aligned}
\|z_m\|^2 &= \langle \sum_{y_m\in Y_m}\langle z_m,y_m\rangle y_m, \sum_{y_m\in Y_m}\langle z_m,y_m\rangle y_m\rangle \\
&= \sum_{y_m\in Y_m}\langle\overline{\langle\langle y_m,z_m\rangle} y_m, \overline{\langle y_m,z_m\rangle} y_m\rangle \\
&= \sum_{y_m\in Y_m}|\langle y_m,z_m\rangle|^2\|y_m\|^2 \\
&= \sum_{y_m\in Y_m}|\langle y_m,z_m\rangle|^2
\end{aligned}
\tag{2.2.17}
$$

Using the same argument and exchanging the orthonormal bases we can show equality.

Then for the proof of (ii), we suppose that $1 \le m \le n$ we choose and fix vectors $y_1 \in Y_1,...,y_{m-1} \in Y_{m-1}, z_{m+1} \in Z_{m+1},...,z_n \in Z_m$. By the Riesz-Frechet theorem there exists the mapping

$$
z \to \psi(A_1y_1,...,A_{m-1}y_{m-1},z,z_{m+1},...,z_n) : \mathcal{K}_m \to \mathbb{C} \tag{2.2.18}
$$

such that it is a bounded linear functional on $\mathcal{K}_m$, and $\exists w \in \mathcal{K}_m$ such that

$$
\psi(A_1y_1,...,A_{m-1}y_{m-1},z,z_{m+1},...,z_n) = \langle z,w\rangle,\ z \in \mathcal{K}_m \tag{2.2.19}
$$

Looking at a particular case, we use Parseval's equation along with Theorem 2.10:

$$\sum_{y_m \in Y_M} |\psi(A_1 y_1, ..., A_{m-1} y_{m-1}, A_m y_m, z_{m+1}, ..., z_n)|^2$$

$$= \sum_{y_m \in Y_M} |\langle A_m y_m, w\rangle|^2$$

$$= \sum_{y_m \in Y_M} |\langle y_m, A_m^* w\rangle|^2$$

$$= \|A_m^* w\|^2. \tag{2.2.20a}$$

Then using Cauchy Schwarz and Theorem 2.10 again we have

$$\|A_m^* w\|^2 \le \|A_m\|^2 \|w\|^2 = \|A_m\|^2 \sum_{z_m \in Z_m} |\langle z_m, w\rangle\rangle|^2$$

$$= \|A_m\|^2 \sum_{z_m \in Z_m} |\psi(A_1 y_1, ..., A_{m-1} y_{m-1}, z, z_{m+1}, ..., z_n)|^2. \tag{2.2.21}$$

In the first line we have expanded the norm square of w in terms of the orthonormal basis Z_m of $\mathcal{K}_m$. Applying this inequality to the whole sum given in (ii) we have

$$\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\phi(y_1, ..., y_{n)}|^2 = \sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\psi(A_1 y_1, ..., A_n y_n)|^2$$

$$\le \|A_n\|^2 \sum_{y_1 \in Y_1} \cdots \sum_{y_{n-1} \in Y_{n-1}} \sum_{z_n \in Z_n} |\psi(A_1 y_1, ..., A_{n-1} y_{n-1}, z_n)|^2$$

$$\le \|A_{n-1}\|^2 \|A_n\|^2$$

$$\times \sum_{y_1 \in Y_1} \cdots \sum_{y_{n-2} \in Y_{n-2}} \sum_{z_{n-1} \in Z_{n-1}} \sum_{z_n \in Z_n} |\psi(A_1 y_1, ..., A_{n-2} y_{n-2}, z_{n-1}, z_n)|^2$$

$$\le \cdots \le \|A_1\|^2 \cdots \|A_n\|^2 \sum_{z_1 \in Z_1} \cdots \sum_{z_n \in Z_n} |\psi(z_1, ..., z_n)|^2 \tag{2.2.22}$$

□

With $\mathcal{H}_1, ..., \mathcal{H}_n$ Hilbert spaces, a mapping $\phi : \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathbb{C}$ is called a Hilbert-Schmidt functional on $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ if it is a bounded multilinear functional, and the sum of Eq(2.2.10) is finite for any choice of the orthonormal bases Y_1 in $\mathcal{H}_1$, ..., Y_n in $\mathcal{H}_n$. Our focus will be on bilinear functionals, however as we have proved the theorem to be true for n-multilinear functionals, it is therefore true for bilinear ones as well. To prevent further confusion we will work with Bilinear functionals.

Theorem 2.12. *If $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, the set $\mathcal{HSF}$ of all Hilbert-Schmidt functionals on $\mathcal{H}_1 \times \mathcal{H}_2$ is itself a Hilbert space when the linear structure, inner product, and norm are defined by*

$$(a\phi + b\psi)(x_1, x_2) = a\phi(x_1, x_2) + b\psi(x_1, x_2) \tag{2.2.23}$$

$$\langle \phi, \psi \rangle = \sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} \phi(y_1, y_2)\overline{\psi(y_1, y_2)} \tag{2.2.24}$$

$$\|\phi\|_2 = \Bigg[\sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} |\phi(y_1, y_2)|^2 \Bigg]^{\frac{1}{2}}, \tag{2.2.25}$$

respectively, where Y_m is an orthonormal basis in $\mathcal{H}_m$, $m = 1, 2$. The sum given by Eq(2.2.24) is absolutely convergent, and the inner product and norm do not depend on the choice of the orthonormal bases Y_1, Y_2.

For each $v(1)$ in $\mathcal{H}_1$, $v(2)$ in $\mathcal{H}_2$ the equation

$$\phi_{v(1),v(2)}(x_1, x_2) = \langle x_1, v(1) \rangle \langle x_2, v(2) \rangle, (x_1 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2) \tag{2.2.26}$$

defines an element $\phi_{v(1),v(2)}$ of $\mathcal{HSF}$, and

$$\langle \phi_{v(1),v(2)}, \phi_{w(1),w(2)} \rangle = \langle w(1), v(1) \rangle \langle w(2), v(2) \rangle \tag{2.2.27}$$

$$\|\phi_{v(1),v(2)}\|_2 = \|v(1)\| \|v(2)\|. \tag{2.2.28}$$

The set $\{\phi_{y(1),y(2)} : y(1) \in Y_1, y(2) \in Y_2\}$ is an orthonormal basis of $\mathcal{HSF}$. There is a unitary transformation U from $\mathcal{HSF}$ onto $l_2(Y_1 \times Y_2)$, such that $U\phi$ is the restriction $\phi|Y_1 \times Y_2$ when $\phi \in \mathcal{HSF}$.

Proof. Choose an orthonormal basis Y_m in $\mathcal{H}_m$, $m = 1, 2$, and then associate with each bounded multilinear functional ϕ on $\mathcal{H}_1 \times \mathcal{H}_2$ the complex-valued function $U\phi$ obtained by restricting ϕ to $Y_1 \times Y_2$. Remember that the condition for a Hilbert-Schmidt functional is that if ϕ is a Hilbert-Schmidt functional if and only if

$$U\phi \in l_2(Y_1 \times Y_2). \tag{2.2.29}$$

If $U\phi = 0$, then

$$\phi(y_1, y_2) = 0 \quad y_1 \in Y_1, y_2 \in Y_2. \tag{2.2.30}$$

Y_m is an orthonormal basis, with its closed linear span being $\mathcal{H}_m$, it follows from the multilinearity and continuity of ϕ that if $U\phi = 0$, for all the basis vectors in the orthonormal bases Y_m, ϕ vanishes throughout $\mathcal{H}_1 \times \mathcal{H}_2$.

Let ϕ, ψ be Hilbert-Schmidt functionals on $\mathcal{H}_1 \times \mathcal{H}_2$, then let $a\phi + b\psi$ be a bounded multilinear functional and $U\phi, U\psi \in l_2(Y_1 \times Y_2)$ thus

$$U(a\phi + b\psi) = aU\phi + bU\psi \in l_2(Y_1 \times Y_2). \tag{2.2.31}$$

This tells us that the linear structure is maintained even under the restriction U. Looking at the inner product, the sum, given by Eq(2.2.24), can be rewritten in the form

$$\langle \phi, \psi \rangle = \sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} \phi(y_1, y_2)\overline{\psi(y_1, y_2)} = \sum_{y \in Y_1 \times Y_2} (U\phi)(y)\overline{(U\psi)(y)} \tag{2.2.32}$$

and it is absolutely convergent with the sum $\langle U\phi, U\psi \rangle$, the inner product in $l_2(Y_1 \times \cdots \times Y_n)$ of the two restricted functionals $U\phi$ and $U\psi$.

The set $\mathcal{HSF}$ of all Hilbert-Schmidt functionals on $\mathcal{H}_1 \times \mathcal{H}_2$is a complex vector space. Eq(2.2.24) then defines an inner product on $\mathcal{HSF}$, the restriction $U|\mathcal{HSF}$ is a one-to-one linear mapping from $\mathcal{HSF}$ into $l_2(Y_1 \times Y_w)$, and $\langle U\phi, U\psi \rangle = \langle \phi, \psi \rangle$ when $\phi, \psi \in \mathcal{HSF}$. The inner product on $l_2(Y_1 \times Y_2)$ is definite, same in $\mathcal{HSF}$; if $\phi \in \mathcal{HSF}$ and $\langle \phi, \phi \rangle = 0$, we have $\langle U\phi, U\phi \rangle = 0$, whence $U\phi = 0 \Rightarrow \phi = 0$. From this we see that $\mathcal{HSF}$ is a pre-Hilbert space, and it is apparent from Eq(2.2.24) that the norm, denoted $\| \|_2$ in $\mathcal{HSF}$ is given by Eq(2.2.25).

$$\begin{aligned}
\langle \phi, \phi \rangle &= \sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} \phi(y_1, y_2)\overline{\phi(y_1, y_2)} \\
&= \left[\sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} |\phi(y_1, y_2)|^2 \right] \\
&= \|\phi\|_2^2 \\
\|\phi\|_2 &= \left[\sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} |\phi(y_1, y_2)|^2 \right]^{\frac{1}{2}}
\end{aligned} \tag{2.2.33}$$

From Theorem 2.11, this norm is independent of the choice of the orthonormal bases $Y_1, .., Y_n$; this is also true of the inner product on $\mathcal{HSF}$.

Now we will show that U brings $\mathcal{HSF}$ onto the whole of the l_2 space. Let $f \in l_2(Y_1 \times Y_2)$ and $x_m \in \mathcal{H}_m, m = 1, 2$, the Cauchy-Schwarz inequality and Parseval equation gives

$$\begin{aligned}
|f(x_1, x_2)| &= |f\left(\sum_{y_1 \in Y_1} \langle x_1, y_1 \rangle y_1, \sum_{y_2 \in Y_2} \langle x_2, y_2 \rangle y_2 \right)| \\
&= \sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} |f(y_1, y_2)\langle x_1, y_1 \rangle \langle x_2, y_2 \rangle|
\end{aligned}$$

$$\leq \left[\sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} |f(y_1, y_2)|^2 \right]^{\frac{1}{2}} \times \left[\sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} |\langle x_1, y_1 \rangle|^2 |\langle x_2, y_2 \rangle|^2 \right]^{\frac{1}{2}}$$

$$= \|f\| \left(\sum_{y_1 \in Y_1} |\langle x_1, y_1 \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{y_2 \in Y_2} |\langle x_2, y_2 \rangle|^2 \right)^{\frac{1}{2}}$$

$$= \|f\| \|x_1\| \|x_2\| \qquad (2.2.34)$$

From this, the equation

$$\phi(x_1, x_2) = \sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} f(y_1, y_2) \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle \qquad (2.2.35)$$

defines a bounded multilinear functional ϕ on $\mathcal{H}_1 \times \mathcal{H}_2$, with $\|\phi\| \leq \|f\|$. The orthonormality of the sets Y_1, Y_2, leads to the fact that

$$(U\phi)(y_1, y_2) = \phi(y_1, y_2) = f(y_1, y_2) \quad y_1 \in Y_1, y_2 \in Y_2 \qquad (2.2.36)$$

so $U\phi = f$. Furthermore $\phi \in \mathcal{HSF}$ because $U\phi \in l_2(Y_1 \times Y_2)$, since U carries $\mathcal{HSF}$ onto the l_2 space.

Since U is a norm preserving linear mapping from $\mathcal{HSF}$ onto $l_2(Y_1 \times Y_2)$ completeness of the l_2 space entails completeness of $\mathcal{HSF}$; so $\mathcal{HSF}$ is a HIlbert space, and U is a unitary operator.

When $v(1) \in \mathcal{H}_1, v(2) \in \mathcal{H}_2$, and $\phi_{v(1),v(2)}$ is a multilinear functional on $\mathcal{H}_1 \times \mathcal{H}_2$ as defined above, it is bounded since

$$|\phi_{v(1),v(2)}(x_1, x_2)| \leq \|v(1)\| \|v(2)\| \|x_1\| \|x_2\| \qquad (2.2.37)$$

by the Cauchy-Schwarz inequality. Furthermore, Parseval's equation gives

$$\langle \phi_{v(1),v(2)}, \phi_{v(1),v(2)} \rangle = \sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} \phi(y_1, y_2) \overline{\phi(y_1, y_2)}$$

$$= \sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} |\phi_{v(1),v(2)}(y_1, y_2)|^2$$

$$= \sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} |\langle y_1, v(1) \rangle|^2 |\langle y_2, v(2) \rangle|^2$$

$$= \left(\sum_{y_1 \in Y_1} |\langle y_1, v(1) \rangle|^2 \right) \left(\sum_{y_2 \in Y_2} |\langle y_2, v(2) \rangle|^2 \right)$$

$$= \|v(1)\|^2 \|v(2)\|^2. \qquad (2.2.38)$$

Hence $\phi_{v(1),v(2)} \in \mathcal{HSF}$ and $\|\phi_{v(1),v(2)}\|_2 = \|v(1)\| \|v(2)\|$. Using Parseval's equation again and by absolute convergence,

$$
\begin{aligned}
&\langle \phi_{v(1),v(2)}, \phi_{w(1),w(2)} \rangle \\
&= \sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} \phi_{v(1),v(2)}(y_1, y_2) \overline{\phi_{w(1),w(2)}(y_1, y_2)} \\
&= \sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} \langle y_1, v(1) \rangle \langle y_2, v(2) \rangle \langle w(1), y_1 \rangle \langle w(2), y_2 \rangle \\
&= \Big(\sum_{y_1 \in Y_1} \langle w(1), y_1 \rangle \langle y_1, v(1) \rangle \Big) \Big(\sum_{y_2 \in Y_2} \langle w(2), y_2 \rangle \langle y_2, v(2) \rangle \Big) \\
&= \langle w(1), v(1) \rangle \langle w(2), v(2) \rangle
\end{aligned} \tag{2.2.39}
$$

To show that $\{\phi_{y(1),y(2)} : y(1) \in Y_1, y(2) \in Y_2\}$ is an orthonormal basis, when $y(1) \in Y_1, y(2) \in Y_2$, the orthonormality of Y_1, Y_2 implies that $U\phi_{y(1),y(2)}$ is the function that takes the value 1 at $(y(1), y(2))$ and 0 elsewhere on $Y_1 \times Y_2$. Therefore

$$\{U\phi_{y(1),y(2)} : y(1) \in Y_1, y(2) \in Y_2\} \tag{2.2.40}$$

is an orthonormal basis of $l_2(Y_1 \times Y_2)$, and therefore

$$\{\phi_{y(1),y(2)} : y(1) \in Y_1, y(2) \in Y_2\} \tag{2.2.41}$$

is a basis of $\mathcal{HSF}$.

□

A useful aspect that arises is that a subset of a Hilbert space is linearly independent, orthogonal, or orthonormal, or an orthonormal basis of that space, if and only if it has the same property relative to the conjugate Hilbert space. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and T is a mapping from the set $\mathcal{H}_1$ into the set $\mathcal{H}_2$, the linearity of $T : \mathcal{H}_1 \to \mathcal{H}_2$ is equivalent to linearity of $T : \bar{\mathcal{H}}_1 \to \bar{\mathcal{H}}_2$, and corresponds to conjugate-linearity of $T : \mathcal{H}_1 \to \bar{\mathcal{H}}_2$ and of $T : \bar{\mathcal{H}}_1 \to \mathcal{H}_2$. Of course, continuity of T is the same in all four situations, when T is linear the operators have the same bound, since the norm on $\mathcal{H}_j$ is the same as that on $\bar{\mathcal{H}}_j$.

Definition 10. *Suppose that $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{K}$ are Hilbert spaces and L is a mapping from $\mathcal{H}_1 \times \mathcal{H}_2$ into $\mathcal{K}$. L is a* ***bounded bilinear mapping*** *if it is linear in each of its variables and there is a real number c such that*

$$\|L(x_1, x_2)\| \leq c\|x_1\| \; \|x_2\|. \tag{2.2.42}$$

The least such constant c is denoted by $\|L\|$.

By a weak Hilbert-Schmidt mapping from $\mathcal{H}_1 \times \mathcal{H}_2$ into $\mathcal{K}$, we mean a bounded multilinear mapping L with the properties:

(i)

$$L_u(x_1, x_2) = \langle L(x_1, x_2), u\rangle \quad \forall u \in \mathcal{K} \tag{2.2.43}$$

where L_u is a Hilbert-Schmidt functional on $\mathcal{H}_1 \times \mathcal{H}_2$.

(ii) There is a real number d such that $\|L_u\|_2 \leq d\|u\|$ for each $u \in \mathcal{K}$.

When these conditions are fulfilled, the least possible value of the constant d is denoted by $\|L\|_2$.

A bounded bilinear mapping $L : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{K}$ is (jointly) continuous relative to the norm topologies on the Hilbert spaces. We see that condition (ii) actually follows from (i) by an application of the closed graph theorem to the mapping.

Theorem 2.13. *Suppose that $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces.*

(i) There is Hilbert space $\mathcal{H}$ and weak Hilbert-Schmidt mapping $p : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}$ with the following property: given any weak Hilbert-Schmidt mapping L from $\mathcal{H}_1 \times \mathcal{H}_2$ into a Hilbert space $\mathcal{K}$, there is a unique bounded linear mapping T from $\mathcal{H}$ into $\mathcal{K}$, such that $L = Tp$; moreover, $\|T\| = \|L\|_2$.

(ii) If $\mathcal{H}'$ and p' have the properties attributed in (i) to $\mathcal{H}$ and p, there is a unitary transformation U from $\mathcal{H}$ onto $\mathcal{H}'$ such that $p' = Up$.

(iii) If $v_m, w_m \in \mathcal{H}_m$ and Y_m is an orthonormal basis of $\mathcal{H}_m, m = 1, 2$, then

$$\langle p(v_1, v_2), p(w_1, w_2)\rangle = \langle v_1, w_1\rangle\langle v_2, w_2\rangle, \tag{2.2.44}$$

the set $\{p(y_1, y_2) : y_1 \in Y_1, y_2 \in Y_2\}$ is an orthonormal basis of $\mathcal{H}$, and $\|p\|_2 = 1$.

Proof. Let $\bar{\mathcal{H}}_m$ be the conjugate Hilbert space of $\mathcal{H}_m$ and let $\mathcal{H}$ be the set of all Hilbert-Schmidt functionals on $\bar{\mathcal{H}}_1 \times \bar{\mathcal{H}}_2$ with the Hilbert space structure given in Theorem 2.12. When $v(1) \in \mathcal{H}_1, v(2) \in \mathcal{H}_2$, let $p(v(1), v(2))$ be the Hilbert-Schmidt functional $\phi_{v(1),v(2)}$ defined on the cartesian product of $\bar{\mathcal{H}}_1 \times \bar{\mathcal{H}}_2$ by

$$\begin{aligned}\phi_{v(1),v(2)}(x_1, x_2) &= \langle x_1, v(1)\rangle^-\langle x_2, v(2)\rangle^- \\ &= \langle v(1), x_1\rangle\langle v(2), x_2\rangle\end{aligned} \tag{2.2.45}$$

Let Y_j be an orthonormal basis of $\mathcal{H}_j, j = 1, 2$ Theorem 2.12 then says that the set $\{p(y_1, y_2) : y_1 \in Y_1, y_2 \in Y_2\}$ is an orthonormal basis of $\mathcal{H}$, and that

$$\begin{aligned}\langle p(v_1, v_2), p(w_1, w_2)\rangle &= \sum_{y_1\in Y_1}\sum_{y_2\in Y_2} \phi_{v_1,v_2}(y_1, y_2)\overline{\phi_{w_1,w_2}(y_1, y_2)} \\ &= \sum_{y_1\in Y_1}\sum_{y_2\in Y_2} \langle v_1, y_1\rangle\langle v_2, y_2\rangle\langle y_1, w_1\rangle\langle y_2, w_2\rangle \\ &= \langle v_1, w_1\rangle\langle v_2, w_2\rangle, \qquad (2.2.46a)\\ \|p(v_1, v_2)\|_2 &= \|v_1\|\|v_2\|. \qquad (2.2.46b)\end{aligned}$$

Note that

$$\|p\|_2 = \|p(y_1, y_2)\|_2 = \|y_1\|\|y_2\| = 1 \tag{2.2.47}$$

From above we know $p : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}$ is a bounded multilinear mapping: we shall prove next that it is a weak Hilbert-Schmidt mapping. Suppose that $\phi \in \mathcal{H}$, and consider the bounded multilinear functional $p_\phi : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$ defined by

$$p_\phi(x_1, x_2) = \langle p(x_1, x_2), \phi \rangle. \tag{2.2.48}$$

With $y(1) \in Y_1, y(2) \in Y_2$, orthonormality of the bases implies that $\phi_{y(1),y(2)}$ takes the value 1 at $(y(1), y(2))$ and 0 elsewhere on $Y_1 \times Y_2$. Thus

$$\begin{aligned} p_\phi(y(1), y(2)) = \langle p(y(1), y(2)), \phi \rangle &= \langle y(1), \phi \rangle \langle y(2), \phi \rangle \\ &= \langle \phi, y(1) \rangle^- \langle \phi, y(2) \rangle^- = \langle \phi_{y(1),y(2)}, \phi \rangle \\ &= \sum_{y_1 \in Y_1} \sum_{y_2 \in Y_2} \phi_{y(1),y(2)}(y_1, y_2) \overline{\phi(y_1, y_2)} \\ &= \overline{\phi(y(1), y(2))}, \end{aligned} \tag{2.2.49}$$

then

$$\sum_{y(1) \in Y_1} \sum_{y(2) \in Y_2} |p_\phi(y(1), y(2))|^2 = \|\phi\|_2^2. \tag{2.2.50}$$

From this we have proven that p_ϕ is a Hilbert-Schmidt functional on $\mathcal{H}_1 \times \mathcal{H}_2$ and that $\|p_\phi\|_2 = \|\phi\|_2$; so $p : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}$ is a weak Hilbert-Schmidt mapping with $\|p\|_2 = 1$.

Next, suppose that L is a weak Hilbert-Schmidt mapping from $\mathcal{H}_1 \times \mathcal{H}_2$ into another Hilbert space $\mathcal{K}$. Let $u \in \mathcal{K}$ and L_u is the Hilbert-Schmidt functional given in Definition 10, while $\phi \in \mathcal{H}$ and let $\mathbb{F}$ be a finite subset of $Y_1 \times Y_2$ then we have using the Cauchy-Schwarz inequality

$$\begin{aligned} &|\langle \sum_{(y_1,y_2) \in \mathbb{F}} \phi(y_1, y_2) L(y_1, y_2), u \rangle| \\ &\leq \sum_{(y_1,y_2) \in \mathbb{F}} |\phi(y_1, y_2)||L_u(y_1, y_2)| \\ \leq \Bigg[\sum_{(y_1,y_2) \in \mathbb{F}} |\phi(y_1, y_2)|^2 \Bigg]^{\frac{1}{2}} &\Bigg[\sum_{(y_1,y_2) \in \mathbb{F}} |L_u(y_1, y_2)|^2 \Bigg]^{\frac{1}{2}} \\ &\leq \|L_u\|_2 \Bigg[\sum_{(y_1,y_2) \in \mathbb{F}} |\phi(y_1, y_2)|^2 \Bigg]^{\frac{1}{2}} \end{aligned}$$

$$\leq \|u\| \|L\|_2 \left[\sum_{(y_1,y_2)\in\mathbb{F}} |\phi(y_1,y_2)|^2 \right]^{\frac{1}{2}} \tag{2.2.51}$$

Hence

$$\| \sum_{(y_1,y_2)\in\mathbb{F}} \phi(y_1,y_2) L(y_1,y_2) \|$$

$$\leq \|L\|_2 \left[\sum_{(y_1,y_2)\in\mathbb{F}} |\phi(y_1,y_2)|^2 \right]^{\frac{1}{2}} \tag{2.2.52}$$

Since

$$\sum_{y_1\in Y_1} \sum_{y_2\in Y_2} |\phi(y_1,y_2)|^2 = \|\phi\|_2^2 < \infty \tag{2.2.53}$$

it then follows from Eq(2.2.52) and the Cauchy criterion that the, unordered, sum

$$\sum_{y_1\in Y_1} \sum_{y_2\in Y_2} \phi(y_1,y_2) L(y_1,y_2) \tag{2.2.54}$$

converges to an element $T\phi \in \mathcal{K}$, and $\|T\phi\| \leq \|L\|_2 \|\phi\|_2$. Thus T is a bounded linear operator from $\mathcal{H}$ into $\mathcal{K}$, and $\|T\| \leq \|L\|_2$. When $y(1) \in Y_1, y(2) \in Y_2$, we have

$$Tp(y_1,y_2) = T\phi_{y(1),y(2)}$$

$$= \sum_{y_1\in Y_1} \sum_{y_2\in Y_2} \phi_{y(1),y(2)}(y_1,y_2) L(y_1,y_2)$$

$$= L(y(1),y(2)). \tag{2.2.55}$$

Both L and Tp are bounded and multiline and Y_m has closed linear span, $\mathcal{H}_m, m = 1,2$ then it follows that $L = Tp$.

The condition that $Tp = L$ uniquely determines the bounded linear operator T, because the range of p contains the orthonormal basis $p(Y_1 \times Y_w)$ of $\mathcal{H}$. $\forall u \in \mathcal{K}$, Parseval's equation gives

$$\|L_u\|_2^2 = \sum_{y_1\in Y_1} \sum_{y_n\in Y_2} |\langle L(y_1,y_2), u\rangle|^2$$

$$= \sum_{y_1\in Y_1} sum_{y_2\in Y_2} |\langle Tp(y_1,y_2), u\rangle|^2$$

$$= \sum_{y_1\in Y_1} \sum_{y_2\in Y_2} |\langle p(y_1,y_2), T^* u\rangle|^2$$

$$= \|T^* u\|^2 \leq \|T\|^2 \|u\|^2; \tag{2.2.56}$$

so we have $\|L\|_2 \leq \|T\|$, and thus $\|L\|_2 = \|T\|$.

We now prove part (ii) of the theorem. Suppose that $\mathcal{H}'$ and $p' : \mathcal{H}_1 \times \times \mathcal{H}_2 \to \mathcal{H}'$ have the properties given in (i). When we have that $\mathcal{K} = \mathcal{H}'$ and $L = p'$, the equation $L = Tp'$ is satisfied when T is the identity operator on $\mathcal{H}'$, and also when T is the projection from $\mathcal{H}'$ onto the closed subspace $[p'(\mathcal{H}_1 \times \mathcal{H}_2)]$ generated by the range $p'(\mathcal{H}_1 \times \mathcal{H}_2)$ of p'. From the uniqueness of T,

$$[p'(\mathcal{H}_1 \times \mathcal{H}_2)] = \mathcal{H}' \tag{2.2.57}$$

furthermore,

$$\|p'\|_2 = \|L\|_2 = \|T\| = \|I\| = 1. \tag{2.2.58}$$

Using a similar argument and letting $\mathcal{K} = \mathcal{H}'$ and $L = p'$, it follows from the properties of the Hilbert space $\mathcal{H}$ and p given in (i), that theres a bounded linear operator $U : \mathcal{H} \to \mathcal{H}$" such that $p' = Up$ and

$$\|u\| = \|L\|_2 = \|p'\|_2 = 1 \tag{2.2.59}$$

The roles of $\mathcal{H}, p$ and $\mathcal{H}', p'$ can be reversed in this argument, so there is a bounded linear operator U' from $\mathcal{H}'$ into $\mathcal{H}$ such that $p = U'p'$ and $\|U'\| = 1$. Since

$$U'Up(x_1, x_2) = U'p'(x_1, x_2) = p(x_1, x_2), \quad \forall x_1 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2, \tag{2.2.60}$$

while

$$[p(\mathcal{H}_1 \times \mathcal{H}_2)] = \mathcal{H} \tag{2.2.61}$$

it follows that $U'U$ is the identity operator on $\mathcal{H}$; and similarly UU' is the identity operator on $\mathcal{H}'$. Finally,

$$\|x\| = \|U'Ux\| \leq \|Ux\| \leq \|x\|, \quad \forall x \in \mathcal{H} \tag{2.2.62}$$

so $\|Ux\| = \|x\|$, and U is an isomorphism from $\mathcal{H}$ onto $\mathcal{H}'$.

□

From part (ii) of the Theorem above, the Hilbert space $\mathcal{H}$ and the multilinear mapping p is uniquely determined by the universal property in part (i). From our definition of a tensor product $\mathcal{H}$ is the tensor product of $\mathcal{H}_1 \otimes \mathcal{H}_2$ and p is the canonical mapping from the cartesian product of $\mathcal{H}_1 \times \mathcal{H}_2$ into $\mathcal{H}_1 \otimes \mathcal{H}_2$. We denote the vector given by $p(x_1, x_2) \in \mathcal{H}_1 \otimes \mathcal{H}_2$ as $x_1 \otimes x_2$. Finite linear combinations of these elementary tensors $x_1 \otimes x_2$ constitute an everywhere-dense subspace of the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. If we let Y_m be an orthonormal basis of $\mathcal{H}_m$ for $m = 1, 2$ then the set

$$\{y_1 \otimes y_2 | y_1 \in Y_1, y_2 \in Y_2\} \tag{2.2.63}$$

is an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$. It is then easily seen that the dimension of the tensor product is the product of the dimensions of each individual Hilbert space

$$\dim(\mathcal{H}_1 \otimes \mathcal{H}_2) = \dim(\mathcal{H}_1)\dim(\mathcal{H}_2). \tag{2.2.64}$$

The vector, or tensor, as a result of the Theorem, acts in a similar fashion to a formal product of x_1, x_2:

$$x_1 \otimes ax_2 = ax_1 \otimes x_2 = a(x_1 \otimes x_2) \tag{2.2.65a}$$
$$x_1 \otimes (x_2' + x_2'') = x_1 \otimes x_2' + x_1 \otimes x_2'' \tag{2.2.65b}$$
$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle \tag{2.2.65c}$$
$$\|x_1 \otimes x_2\| = \|x_1\|\|x_2\| \tag{2.2.65d}$$

3 Operator Spaces

The study of the tensor product of Hilbert spaces resulting in a Hilbert space asks the question as what operators can act on these tensors. Prior the tensor product, we would use the bounded operators on each Hilbert space, however this cannot be the case for the tensor product. We will consider the tensor product of two operator spaces and show that it is an operator space. This new operator space will be the bounded operators on the tensor product of two Hilbert Spaces. My methodology follows the description given by Pisier's *Introduction to Operators Space Theory* and Stinespring's Dilation theorem is given by Vern Paulsen's textbook *Completely Bounded Maps and Operator Algebras*. My contribution to this section is showing the details of the proofs.[5] [6]

3.1 Completely Bounded Maps

To discuss operator spaces we must first define a few terms.

Definition 11. *A C^** **- *algebra*** *is a Banach $*$-algebra satisfying the identity*

$$\|x^*x\| = \|x\|^2 \tag{3.1.1}$$

for any element x in the algebra.

We can consider the space of all bounded operators $B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ equipped with the operator norm to be a C^*-algebra. We may also consider an closed subspace

$$S \subset B(\mathcal{H}) \tag{3.1.2}$$

a C^*-algebra if it is stable under product and involution. Now let us define an operator space.

Definition 12. *An **operator space** is a closed subspace of $B(\mathcal{H})$.*

In other words we can think of operator spaces as closed subspace of the C^*-algebras. To further understand what an operator space is we can consider them to be Banach spaces X equipped with an extra structure in the form of an isometric embedding into the space of all bounded operators on a Hilbert space $B(\mathcal{H})$

$$X \subset B(\mathcal{H}). \tag{3.1.3}$$

Nevertheless, because of the isometric embedding into the space of bounded operators on a Hilbert space, the morphisms from one operator space to another requires them to be completely bounded maps. Concretely, suppose we have E, F as Banach spaces then we can view them as operator spaces through the embedding

$$E \subset C(B_{E^*}), \quad F \subset C(B_{F^*}). \tag{3.1.4}$$

Let us now define completely bounded maps.

Definition 13. *Let $E \subset B(\mathcal{H})$ and $F \subset B(\mathcal{H})$ be operator spaces and consider a map $u : E \to F$. For any $n \geq 1$, let*

$$M_n(E) = \{(x_{ij})_{ij \leq n} | x_{ij} \in E\} \tag{3.1.5}$$

be the space of $n \times n$ matrices with entries in E. In particular we have a natural identification

$$M_n(B(\mathcal{H})) \simeq B(\ell_2^n(\mathcal{H})), \tag{3.1.6}$$

where $\ell_2^n(\mathcal{H})$ means $\underbrace{\mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n \text{ times}}$. Thus, we may equip $M_n(B(\mathcal{H}))$ and a fortiori its subspace

$$M_n(E) \subset M_n(B(\mathcal{H})) \tag{3.1.7}$$

with the norm induced by

$$B(\ell_2^n(\mathcal{H})). \tag{3.1.8}$$

Then, for any $n \geq 1$, the linear map $u : E \to F$ allows us to define a linear map

$$u_n : M_n(E) \to M_n(F) \tag{3.1.9}$$

defined by

$$u_n \begin{pmatrix} & \vdots & \\ \cdots & x_{ij} & \cdots \\ & \vdots & \end{pmatrix} = \begin{pmatrix} & \vdots & \\ \cdots & u(x_{ij}) & \cdots \\ & \vdots & \end{pmatrix}. \tag{3.1.10}$$

Definition 14. *A map $u : E \to F$ is called **completely bounded** if*

$$\sup_{n\geq 1} \|u_n\|_{M_n(E)\to M_n(F)} < \infty. \tag{3.1.11}$$

We define

$$\|u\|_{cb} = \sup_{n\geq 1} \|u_n\|_{M_n(E)\to M_n(F)}. \tag{3.1.12}$$

and we denote by $CB(E,F)$ the Banach space of all completely bounded maps from E to F equipped with the completely bounded norm $\|.\|_{cb}$.

We defined $CB(E,F)$ because we will use this to replace $B(E,F)$ of all bounded operators from E to F as we stated before that we need to use completely bounded maps from one operator space to another.

The composition of two completely bounded maps is also completely bounded. Let $G \subset B(L)$ and let $v : F \to G$ be a completely bounded map. The composition given as $uv : E \to G$ is completely bounded and

$$\|uv\|_{cb} \leq \|v\|_{cb}\|u\|_{cb}. \tag{3.1.13}$$

Take note that clearly the space of completely bounded maps from E to F, $CB(E,F)$, is a subspace of all bounded operators from E to F, $B(E,F)$.

When we have $\|u\|_{cb} \leq 1$ then u is called "completely contractive" or a "complete contraction." Then we replace the notion of isometry with that of a "complete isometry."

Definition 15. *A map $u : E \to F$ is called a **complete isometry** if*

$$u_n : M_n(E) \to M_n(F) \tag{3.1.14}$$

is an isometry for all $n \geq 1$.

Similarly to the definition of completely isometric, a map $u : E \to F$ is called completely positive if $u_n : M_n(E) \to M_n(F)$ is positive for all n. Now let us define completely isomorphic operator spaces.

Definition 16. *Two operator spaces E, F are called **completely isomorphic** if there is al near isomorphism $u : E \to F$ such that u and u^{-1} are completely bounded.*

To give a more concrete understanding of operator spaces we will look at specifically Hilbert spaces. Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces and let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. The mapping

$$x \to \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \tag{3.1.15}$$

is an isometric embedding of $B(\mathcal{H}_1, \mathcal{H}_2)$ into $B(\mathcal{H})$. Therefore we can view $B(\mathcal{H}_1, \mathcal{H}_2)$ as an operator space. The norm induced on the operator space is the norm of the space $B(\ell_2^n(\mathcal{H}_1), \ell_2^n(\mathcal{H}_2))$.

Finally, let us consider the tensor product of two operator spaces.

Theorem 3.1. *The tensor product of two operator spaces exists.*

Proof. The proof of this theorem is analogous to our previous theorem that the tensor product of vector and Hilbert spaces exist. However, we will use completely bounded bilinear functions from the Cartesian product of two operator spaces, $V \times W$, to an operator space denoted as $V \otimes W$ and a completely bounded linear function from $V \otimes W$ to the space X.

□

The algebraic tensor product of two operator spaces is just a vector space, however the norms on the vector space make it an operator space. There is a natural identification of tensor products of bounded linear operators on Hilbert spaces as bounded operators on the tensor product of those Hilbert spaces. Let the algebraic tensor product of two Hilbert spaces $\mathcal{H}, \mathcal{K}$, $\mathcal{H} \otimes \mathcal{K}$ complete with respect to the norm induced by the inner product on the elementary tensors be denoted as $\mathcal{H} \otimes_2 \mathcal{K}$. Then the natural identification between $B(\mathcal{H}) \otimes B(\mathcal{K}) \subseteq B(\mathcal{H} \otimes_2 \mathcal{K})$ is given by

$$(T \otimes S)(x \otimes y) = T(x) \otimes S(y), \quad T \in B(\mathcal{H}), S \in B(\mathcal{K}). \tag{3.1.16}$$

3.2 Minimal Tensor Product

We will now define the minimal tensor product. Pisier defines it by considering two operator spaces $E \subset B(\mathcal{H})$ and $F \subset B(\mathcal{K})$. Their minimal tensor products is the completion of the algebraic tensor product of E and F denoted as $E \otimes F$ with respect to the norm given by $B(\mathcal{H} \otimes_2 \mathcal{K})$ due to the embedding

$$E \otimes F \subset B(\mathcal{H} \otimes_2 \mathcal{K}), \tag{3.2.1}$$

where the subscript 2 indicates the norm on the Hilbert space $\mathcal{H} \otimes_2 \mathcal{K}$. The linear space is then denoted as $E \otimes_{\min} F$ and $\| \ \|_{\min}$ is the norm.

The inner product on the minimal tensor product is given to be

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle \equiv \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle. \tag{3.2.2}$$

There it is also easy to see that the norm of the elementary tensor $e \otimes f$ is then

$$\|e \otimes f\| = \|e\| \|f\|. \tag{3.2.3}$$

Our goal is to show a generalization of the Cauchy-Schwarz inequality but for operators. To begin we must define a few terms and prove a theorem and a few lemmas.

Definition 17. *A C^* - algebra, A, is* ***unital*** *if it admits a multiplicative identity 1 such that*

$$1^*a = (a^*1)^* = a \tag{3.2.4a}$$
$$a1^* = a \tag{3.2.4b}$$
$$\|1\| = \|1^*1| \ = \|1\|^2 \tag{3.2.4c}$$

Definition 18. *A linear algebra homomorphism between C^* - algebras $\pi : A \to B$ which is self-adjoint. ie. $\pi(a^*) = \pi(a)^*$ is called a $*$-**homomorphism**. A **unital** $*$-**homomorphism** is such that $\pi(1_A) = \pi(1_B)$.*

Definition 19. *If B is a C^* -algebra and $\phi : S \to B$ is a linear map, where S is an operator system, then we define $\phi_n : M_n(S) \to M_n(B)$ by $\phi_n((a_{i,j})) = (\phi(a_{i,j}))$. We call ϕ* ***completely positive*** *if ϕ is n-positive for all n.*

Then with these definitions we are now able to prove Stinespring's Dilation Theorem.

Theorem 3.2. *Let $\mathcal{A}$ be a unital C^*- algebra, and let $\phi : \mathcal{A} \to B(\mathcal{H})$ be a completely positive map. Then there exists a Hilbert space $\mathcal{K}$, a unital $*$-homomorphism $\pi : \mathcal{A} \to B(\mathcal{K})$, and a bounded operator $V : \mathcal{H} \to \mathcal{K}$ with $\|\phi(1)\| = \|V\|^2$ such that*

$$\phi(a) = V^*\pi(a)V \tag{3.2.5}$$

Proof. Consider the algebraic tensor product $\mathcal{A} \otimes \mathcal{H}$, of the unital C^*-algebra and the Hilbert Space $\mathcal{H}$, and define a symmetric bilinear function $\langle \cdot, \cdot \rangle$ on this space by setting

$$\langle a \otimes x, b \otimes y \rangle = \langle \phi(b^*a)x, y \rangle_{\mathcal{H}}, \tag{3.2.6}$$

and extending linearly, where $\langle , \rangle_{\mathcal{H}}$ is the inner product on $\mathcal{H}$. From the definition, ϕ is completely positive thereby ensuring that $\langle \cdot, \cdot \rangle$ is positive semidefinite since

$$\left\langle \sum_{j=1}^{n} a_j \otimes x_j, \sum_{i=1}^{n} a_i \otimes x_i \right\rangle = \left\langle \phi_n(a^*a) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle_{\mathcal{H}^{(n)}} \geq 0, \tag{3.2.7}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}^{(n)}}$ denotes the inner product not the direct sum $\mathcal{H}^{(n)}$ of n copies of $\mathcal{H}$, given by

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle_{\mathcal{H}^{(n)}} = \langle x_1, y_1 \rangle_{\mathcal{H}} + \cdots + \langle x_n, y_n \rangle_{\mathcal{H}} \tag{3.2.8}$$

A result of positive semidefinite bilinear forms is that they satisfy the Cauchy-Schwarz inequality,

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle \cdot \langle v, v\rangle \tag{3.2.9}$$

Therefore we have that

$$\{u \in \mathcal{A} \otimes \mathcal{H} | \langle u, u\rangle = 0\} = \{u \in \mathcal{A} \otimes \mathcal{H} | \langle u, v\rangle = 0 \ \ \forall v \in \mathcal{A} \otimes \mathcal{H}\} \tag{3.2.10}$$

is a subspace, $\mathcal{N}$, of $\mathcal{A} \otimes \mathcal{H}$. The induced bilinear form on the quotient space $\mathcal{A} \otimes \mathcal{H}/\mathcal{N}$ defined by

$$\langle u + \mathcal{N}, v + \mathcal{N}\rangle = \langle u, v\rangle \tag{3.2.11}$$

will be an inner product. We let $\mathcal{K}$ denote the Hilbert space that is the completion of the inner product space $\mathcal{A} \otimes \mathcal{H}/\mathcal{N}$.

If $a \in \mathcal{A}$, define a linear map $\pi(a) : \mathcal{A} \otimes \mathcal{H} \to \mathcal{A} \otimes \mathcal{H}$ by

$$\pi(a)\left(\sum a_i \otimes x_i\right) = \sum (aa_i) \otimes x_i \tag{3.2.12}$$

A matrix factorization shows that the following inequality in $M_n(\mathcal{A})^+$ is satisfied:

$$(a_i^* a^* a a_j) \leq \|a^* a\| \cdot (a_i^* a_j), \tag{3.2.13}$$

and consequently,

$$\begin{aligned} &\left\langle \pi(a)\left(\sum a_j \otimes x_j\right), \pi(a)\left(\sum a_i \otimes x_i\right)\right\rangle \\ &= \sum_{i,j} \langle \phi(a_i^* a^* a a_j) x_j, x_i\rangle_{\mathcal{H}} \leq \|a^* a\| \cdot \sum_{i,j} \langle \phi(a_i^* a_j) x_j, x_i\rangle_{\mathcal{H}} \\ &= \|a\|^2 \cdot \left\langle \sum a_j \otimes x_j, \sum a_i \otimes x_i \right\rangle. \end{aligned} \tag{3.2.14}$$

Therefore, $\pi(a)$ leave $\mathcal{N}$ invariant and consequently induces a quotient linear transformation on $\mathcal{A} \otimes \mathcal{H}/\mathcal{N}$, which we will still denote by $\pi(a)$. The above inequality also shows that $\pi(a)$ is bounded with $\|\pi(a)\| \leq \|a\|$. Thus, $\pi(a)$ extends to a bounded linear operator on $\mathcal{K}$, which we will still denote by $\pi(a)$. It is straightforward to verify that the map $\pi : \mathcal{A} \to B(\mathcal{K})$ is a unital $*$ - homomorphism.

Now define $V : \mathcal{H} \to \mathcal{K}$ via

$$V(x) = 1 \otimes x + \mathcal{N} \tag{3.2.15}$$

Then V is bounded, since

$$\|Vx\|^2 = \langle 1 \otimes x, 1 \otimes x\rangle = \langle \phi(1)x, x\rangle_{\mathcal{H}} \leq \|\phi(1)\| \cdot \|x\|^2. \tag{3.2.16}$$

Indeed, it is clear that $\|V\|^2 = \sup\{\langle \phi(1)x, x\rangle_{\mathcal{H}} : \|x\| \leq 1\} = \|\phi(1)\|$.
To complete the proof, we only need to observe that

$$\langle V^*\pi(a)Vx, y\rangle_{\mathcal{H}} = \langle \pi(a)1 \otimes x, 1 \otimes y\rangle_{\mathcal{K}} = \langle \phi(a)x, y\rangle_{\mathcal{H}}, \quad \forall x, y \in \mathcal{H}, \tag{3.2.17}$$

and so $V^*\pi(a)V = \phi(a)$.

□

Let us look at a lemma that utilizes this theorem and gives us the minimal norm of the tensor.

Lemma 3.3. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. We denote $S_2(\mathcal{K}, \mathcal{H})$ the space of all Hilbert-Schmidt operators $x : \mathcal{K} \to \mathcal{H}$ and we denotes its Hilbert-Schmidt norm by $\|x\|_{HS}$. Consider finite sequences (a_1) in $B(\mathcal{H})$ and (b_i) in $B(\mathcal{K})$. Then we have*

$$\|\sum a_i \otimes \bar{b}_i\|_{B(\mathcal{H})\otimes_{\min}\overline{B(\mathcal{K})}} = \sup\{\|\sum a_i x b_i^*\|_{HS} | x \in S_2(\mathcal{K}, \mathcal{H}), \|x\|_{HS} \leq 1\}. \tag{3.2.18}$$

Proof. Remember that there is a natural identification between $B(\mathcal{H}) \otimes_{\min} \overline{B(\mathcal{K})}$ and $B(\mathcal{H} \otimes_2 \bar{\mathcal{K}})$. Then the norm on $B(\mathcal{H}) \otimes_{\min} \overline{B(\mathcal{K})}$ is the norm induced by $B(\mathcal{H} \otimes_2 \bar{\mathcal{K}})$. Now if we identify $\mathcal{H} \otimes_2 \bar{\mathcal{K}}$ with $S_2(\mathcal{H}, \mathcal{H})$ the norm induced is the Hilbert-Schmidt norm. Using Stinespring's dilation theorem we can identify the operator $\sum a_i \otimes \bar{b}_i$ with $x \to \sum a_i x b_i^*$ and the norm of then tensor is

$$\|\sum a_i \otimes \bar{b}_i\|_{B(\mathcal{H})\otimes_{\min}\overline{B(\mathcal{K})}} = \sup\{\|\sum a_i x b_i^*\|_{\mathrm{HS}} | x \in S_2(\mathcal{K}, \mathcal{H}), \|x\|_{\mathrm{HS}} \leq 1\}. \tag{3.2.19}$$

□

We will prove one more lemma before we prove the inequality for operators.

The goal of the thesis was to understand the mathematical reasoning and logic behind the "raising" of scalars to operators in quantum mechanics. In the following theorem we will finally see the justification behind this action.

Theorem 3.4. *Let x_i, y_i be elements of $B(\ell_2)$, the bounded operators on ℓ_2 and let $S_2(\ell_2, \ell_2)$ be the Hilbert-Schmidt operators from ℓ_2 to ℓ_2. Then the minimal norm of $\sum x_i \otimes \bar{y}_i$ is*

$$\|\sum x_i \otimes \bar{y}_i\|_{\min} \leq \|\sum x_i \otimes \bar{x}_i\|_{\min}^{\frac{1}{2}} \|\sum y_i \otimes \bar{y}_i\|_{\min}^{\frac{1}{2}}. \tag{3.2.20}$$

Proof. Using Stinespring's Dilation theorem we are able to rewrite the tensor as an operator from ℓ_2 to ℓ_2 and by definition the norm becomes

$$\left\| \sum x_i \otimes \bar{y}_i \right\|_{\min} \leq \sup \left\{ \left| \sum \langle x_i a y_i^*, b \rangle \right| \right\}, \tag{3.2.21}$$

where the supremum runs over all a, b in the unit ball of $S_2(\ell_2, \ell_2)$. Because the Hilbert-Schmidt operators are of the S2 class of Schatten operators we know that the operator a can be written as $a = a_1 a_2$, with $\mathrm{tr}|a_1|^4 \leq 1$ and $\mathrm{tr}|a_2|^4 \leq 1$, similarly with the operator b; a_1, a_2, b_1, and b_2 are of the S4 Schatten class operators. Furthermore the Hilbert-Schmidt inner product is equivalent to the trace of the operators $\langle A, B \rangle = \mathrm{tr}(AB^*)$ and therefore we have

$$\begin{aligned} \langle x_i a y_i^*, b \rangle &= \mathrm{tr}(x_i a y_i^* b^*) \\ &= \mathrm{tr}(x_i a_1 a_2 y_i^* b_2^* b_1^*) \\ &= \mathrm{tr}(b_1^* x_i a_1 a_2 y_i^* b_2^*) \\ &= \langle b_1^* x_i a_1, b_2 y_i a_2^* \rangle. \end{aligned} \tag{3.2.22}$$

Therefore using the Cauchy-Schwarz inequality we obtain

$$\left| \sum \langle x_i a y_i^*, b \rangle \right| = \left| \sum \langle b_1^* x_i a_1, b_2 y_i a_2^* \rangle \right| \leq \left(\sum \|b_1^* x_i a_1\|_{\mathrm{HS}}^2 \right)^{\frac{1}{2}} \left(\sum \|b_2 y_i a_2^*\|_{\mathrm{HS}}^2 \right)^{\frac{1}{2}}. \tag{3.2.23}$$

Let us now look individually at each term inside the bracket.

$$\begin{aligned} \sum \|b_1^* x_i a_1\|_{\mathrm{HS}}^2 &= \sum \langle b_1^* x_i a_1, b_1^* x_i a_1 \rangle \\ &= \sum \mathrm{tr}(b_1^* x_i a_1 a_1^* x_i^* b_1) \\ &= \sum \mathrm{tr}(x_i a_1 a_1^* x_i^* b_1 b_1^*) \\ &\leq \left\| x_i \otimes \bar{x}_i \right\|_{\min}. \end{aligned} \tag{3.2.24}$$

The last line follows from out first step and the inequality originates from the definition of the norm. Similarly we can do this for the other term and therefore we attain

$$\left\| \sum x_i \otimes \bar{y}_i \right\|_{\min} \leq \left\| \sum x_i \otimes \bar{x}_i \right\|_{\min}^{\frac{1}{2}} \left\| \sum y_i \otimes \bar{y}_i \right\|_{\min}^{\frac{1}{2}}. \tag{3.2.25}$$

□

The idea of quantization usually refers to the idea in quantum mechanics to replace the scalars, real or complex, with operators and then apply commutation relations. Our inequality for operators is really the generalization of the Cauchy-Schwarz inequality for complex numbers. The field scalars are then replaced by elements of $B(\ell_2)$ and the product is therefore replaced by the tensor product.

Conclusion

The goal of this thesis was to understand the mathematical formalism behind quantization in quantum theory. In order to accomplish my aim I studied the tensor product of Hilbert spaces and operator spaces. In doing so, I was able to prove the existence and uniqueness of the tensor product. Furthermore, I was able to complete the tensor product space of operators and complete it with respect to the norm induced by its isometric embedding into the bounded operators on the tensor product of two Hilbert spaces. Finally, using Stinespring's Dilation theorem I was able to prove the inequality for operators.

References

[1] S. Axler, *Linear Algebra Done Right* (New York: Springer, 1996. Print.)

[2] J. Foster, J.D Nightingale, *A Short Course in General Relativity* (New York: Springer, 1995).

[3] N. Young, *An Introduction to Hilbert Spaces* (Cambridge University Press, 1988).

[4] R. V. Kadison and J. R. Ringrose *Fundamentals of the Theory of Operator Algebras* (New York: Academic, 1983. Print.)

[5] V. Paulsen *Completely Bounded Maps and Operator Algebras* (Cambridge: Cambridge UP, 2002. Print.)

[6] G. Pisier *Introduction to Operator Space Theory* (Cambridge, U.K.: Cambridge UP, 2003. Print.)