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# Isometries and Spontaneous Lorentz Violation in General Relativity

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# 1 Abstract

General Relativity, or GR, is a theory which describes gravity as a manifestation of the curvature of space and time. While the other three fundamental forces in nature are represented as field theories, GR is a geometric theory. In the search for a way to reconcile the field theories for the electromagnetic, strong, and weak forces with the gravitational force, a logical place to start is by re-expressing GR as a field theory. In doing so, we find that the theory contains a number of symmetries. When we solve the equations in GR, we find that by choosing certain solutions we break some of the symmetries of the system (through a mechanism known as spontaneous symmetry breaking.) We seek to find out how the total number of symmetries, which are represented mathematically as isometries, change for different solutions when we add a vector potential field which spontaneously breaks (Lorentz) symmetry. Such a mechanism is thought to occur in higher dimensional theories such as String Theory, and so a better understanding of the mechanism in GR could be useful in later work. We will find that the number of symmetries is reduced from 10 to 6 with a time-dependent vacuum solution, which is the same number of symmetries that our actual universe is thought to possess.

## 2 Acknowledgements

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# Contents

## 3 Introduction

In 1916 Albert Einstein published his theory of General Relativity. He developed the theory almost entirely on his own, and without the aid of any of today's technologies which could have helped to verify the theory. General relativity, or GR as it is commonly referred to, is a geometric theory describing gravitation, and which resolves the problems with gravity from classical Newtonian physics. As technologies have arisen to test the predictions of GR, the theory has met them head on and passed with flying colors; it has been proven to be correct time and time again. Instead of treating gravity as a force, as in Newtonian mechanics, gravity is described in general relativity as a manifestation of the curvature of space and time. In the theory the curvature of spacetime is related to mass, energy, and momentum, and is described by a system of partial differential equation (the Einstein Field Equations.) As we will see, there are other ways to express the theory which may be more useful.

The Standard Model is a theory which has arisen in particle physics, a field which seeks to understand the dynamics of particles and how they interact. It is well known that there are four fundamental interactions which particles are subject to: strong, weak, electromagnetic, and gravitational. The Standard Model unites the first three of these, accurately describing the electroweak and quantum chromodynamic interactions between particles. Arguably the largest unresolved issue in modern physics is the inability to merge general relativity, a theory which has been verified experimentally, with quantum mechanics or the Standard Model. There exist, however, clues as to where this unification may arise. The Standard Model describes the other three fundamental interactions as quantum field theories, and so a good place to start the search for a unified theory is by describing GR as a quantum field theory.

To be more specific, we want to express GR as a gauge theory, a field theory which, when described by a Lagrangian, is invariant under a certain group of transformations (what are known as a gauge transformations.) A Lagrangian describes the dynamics of a system, and thus we can use it to express Einstein's GR, using scalars, vectors, and tensors to express particles. We will find that in GR the gauge transformations, the mathematical transformations which leave the Lagrangian invariant, are actually a group of transformations known as diffeomorphisms. It is the belief of many physicists that a more complete understanding of diffeomorphism invariance in GR could provide hints of how to unify a theory of gravity with theories for the other three fundamental interactions. It will be beyond the scope of this thesis to suggest ways in which gravitation may be related to the other interactions; instead, we will examine certain aspects of the diffeomorphism invariance in GR.

We will start this thesis by looking at field theories, and seeing how it is possible to express GR in this way. In studying field theories, we will look at the Standard Model, and some of its features. One of these features will be the spontaneous symmetry breaking mechanism. We will look at ground state, or vacuum, solutions in GR, and see how spontaneously choosing a specific vacuum solution can result in the loss of symmetries of the theory. By looking at how mechanisms that occur in other field theories occur in GR, we can hopefully better our understanding of the intricacies of General Relativity.

The theory of general relativity is an extremely technical theory, and forces us to use some mathematical machinery which may not be necessary in other fields of physics. We must learn how to understand scalars, vectors, and tensors in the context of manifolds, which are abstract spaces used, in GR, to describe the curvature of spacetime. In this thesis, we will take the convention that the Minkowski metric for flat spacetime is:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

Working outside of a flat space, we will be forced to abandon our definition of a vector as an object pointing from one location to another. Instead we will have to generalize the idea of the vector to an object which exists at a certain location. We will also have to generalize the idea of the derivative when working in a curved space. In this thesis tensors will be used frequently, and in order to simplify notation, we will assume that the reader has a working knowledge of the Einstein summation convention. We will also assume, for simplicity, that $c = \hbar = 1$, where c is the speed of light and $\hbar$ is Planck's constant. Once we are better able to understand the mathematics of manifolds, we will be able to study diffeomorphisms.

By better understanding manifolds, we will also be able to look at symmetries. We will find that symmetries of the metric, the mathematical object which helps to characterize the manifold, are called isometries. It will eventually be possible to answer the question, "How does one find the isometries of a space?" We will then be able to investigate the symmetries of GR. We will first look at symmetries in a standard GR model, then look at the isometries of a Bumblebee Model, which will be the Einstein-Maxwell theory with a vector potential field. Since spontaneously choosing a vacuum solution for the vector potential field will break Lorentz symmetry, we will be able to address the question, "How are the usual isometries of general relativity affected by spontaneous Lorentz breaking?"

As should be apparent to the reader, much of this thesis will be focused on building up the mathematical formalities used to study GR. We will look at field theories, gauge theories, the standard model, manifolds, diffeomorphisms, isometries, and spontaneous symmetry breaking. Once we are able to understand all of these concepts, we will be able to put them together in order to look at how isometries in GR are affected by spontaneous breaking of Lorentz symmetry. We will start by first looking at classical Lagrangian field theory.

## 4 Classical Lagrangian Field Theory

We will begin our exploration with a look at classical field theory. For any given model, the dynamics of the system can be described by a Lagrangian. This Lagrangian can then be used to find the equations of motion as described for that model. If we let T be the kinetic energy of a system and V be the potential energy of a system, the Lagrangian, $\mathcal{L}$, is then

$$\mathcal{L} = T - V \tag{4.1}$$

If we take the integral of the action (with respect to a four-dimensional spacetime,) the result is called the action, S, such that

$$S = \int \mathcal{L}\, d^4x \tag{4.2}$$

If we take a small variation in the action, we know by the principle of least action that it should be equal to zero.

$$\delta S = 0 \tag{4.3}$$

Combining equations (4.2) and (4.3) gives that

$$\delta S = \int \delta\mathcal{L}\, d^4x = 0 \tag{4.4}$$

This Lagrangian formulation lends itself well to describing particles; the particle can be thought of as a field, and can thus be written in terms of a Lagrangian. If we are able to write out the Lagrangian for the particle, then we simply have to vary the Lagrangian, integrate it, and set it equal to zero in order to find the equations of motion for the particle. Let's look at a simple example in order to show the significance and power of the Lagrangian.

### 4.1 Using The Lagrangian (A Simple Example)

Consider a particle moving in one dimension. This particle has kinetic and potential energy

$$\begin{aligned} T &= \tfrac{1}{2}m\dot{x}^2 \\ V &= V(x) \end{aligned} \tag{4.5}$$

Now consider a change in $x$ such that $x \to \delta x$. Then

$$\begin{aligned} \delta\mathcal{L} &= \delta(\tfrac{1}{2}m\dot{x}^2) - \delta V(x) \\ &= m\dot{x}\delta\dot{x} - \tfrac{dV(x)}{dx}\delta x \end{aligned} \tag{4.6}$$

Then we can say that $\delta\dot{x} = \frac{d}{dt}(\delta x)$, so that

$$
\begin{aligned}
\delta\mathcal{L} &= m\dot{x}\tfrac{d}{dx}(\delta x) - \tfrac{dV(x)}{dx}(\delta x) \\
&= m\tfrac{d}{dt}(\dot{x}\delta x) - \ddot{x}\delta x - \tfrac{dV(x)}{dx}\delta x
\end{aligned}
\tag{4.7}
$$

If we then look at $\frac{d}{dt}(\dot{x}\delta x) = \ddot{x}\delta x + \dot{x}\frac{d}{dt}\delta x$, we can use equation (2.4), integrating from time a to time b. Then

$$
\begin{aligned}
\delta S &= \int \delta\mathcal{L}\, d^4x = 0 \\
&= \int_a^b \tfrac{d}{dt}(\dot{x}\delta x)\, dt - \int_a^b (m\ddot{x} + \tfrac{dV(x)}{dx})\delta x\, dt = 0
\end{aligned}
\tag{4.8}
$$

On the left-hand term, we see that we get

$$
\begin{aligned}
\dot{x}\delta x|_a^b &= 0 \\
\Rightarrow \delta x(a) &= \delta x(b) = 0
\end{aligned}
\tag{4.9}
$$

And for the right-hand term, we get that

$$
m\ddot{x} + \tfrac{dV(x)}{dx} = 0 \tag{4.10}
$$

We know that $\ddot{x} = a$, the acceleration of the particle. We can then deduce that the negative of the second term ($F = -\frac{dV(x)}{dx}$), is the negative of the force on the particle. Thus we have obtained Newton's second law

$$
F = ma \tag{4.11}
$$

Clearly, the Lagrangian formulation is a powerful tool.

## 4.2 Classical Scalar Field

Consider a spinless massive free particle. We know that the four-momentum for the particle can be written

$$
p^\mu = \hbar k^\mu = (\tfrac{E}{c}, \vec{p}) \tag{4.12}
$$

Remembering that we're setting $\hbar = c = 1$ for simplicity, we get that

$$
p^\mu = k^\mu = (E, \vec{p}) \tag{4.13}
$$

We can also note that

$$
k_\mu = \eta_{\mu\nu}k^\nu = (-E, \vec{p}) \tag{4.14}
$$

Let's now see that

$$k_\mu k^\mu = -E^2 + \vec{p}^2 \tag{4.15}$$

$$E^2 = \vec{p}^2 - k_\mu k^\mu \tag{4.16}$$

We also know, however, that

$$E^2 = m^2c^4 + \vec{p}^2c^2 = m^2 + \vec{p}^2 \tag{4.17}$$

And comparing (4.16) and (4.17) we see that for a massive free particle it must hold that

$$k_\mu k^\mu = -m^2 \tag{4.18}$$

It is then also true that for any massless free scalar particle (like a photon, for example) that $k_\mu k^\mu = 0$.

Now we'll claim that a massive free particle is described by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \tfrac{1}{2}m^2\phi^2 \tag{4.19}$$

where $\phi$ is the scalar field, m is the mass, and $\partial_\mu$ is the usual shorthand notation for $\frac{\partial}{\partial x^\mu}$.

Using the same method as in the previous section, it is straightforward to solve for the equations of motion. First we must vary the Lagrangian with respect to $\phi$. We'll let $\phi \to \delta\phi(x)$.

$$\delta\mathcal{L} = \tfrac{1}{2}(\partial_\mu \delta\phi)(\partial^\mu \phi) + \tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \delta\phi) - m^2\phi\delta\phi \tag{4.20}$$

We can rewrite the first term of (4.20) as

$$\tfrac{1}{2}(\partial_\mu \delta\phi)(\partial^\mu \phi) = \tfrac{1}{2}\partial_\mu(\delta\phi\partial^\mu\phi) - \tfrac{1}{2}\delta\phi(\partial_\mu\partial^\mu\phi) \tag{4.21}$$

We know that eventually we will put the varied Lagrangian inside of an integral. When the first term of (4.21) appears in the integral, it will go to zero since $\delta\phi = 0$ at the boundaries. Thus we can write the functional varied Lagrangian as

$$\delta\mathcal{L} = -\tfrac{1}{2}\delta\phi(\partial_\mu\partial^\mu\phi) - \tfrac{1}{2}\delta\phi(\partial^\mu\partial_\mu\phi) - m^2\phi\delta\phi \tag{4.22}$$

And since we know that $\partial_\mu\partial^\mu = \partial^\mu\partial_\mu$ we can combine the first two terms of (4.22) and pull out an overall $-\delta\phi$ and write that

$$\delta\mathcal{L} = -\delta\phi(\partial_\mu\partial^\mu\phi + m^2\phi) \tag{4.23}$$

From (2.4) and the principle of least action, we know that (4.23) must be equal to zero. We can also replace $\partial_\mu\partial^\mu$ with D'Alembertian operator $\Box$.

$$(\Box + m^2)\phi = 0 \tag{4.24}$$

We can see (and will show, for those who don't trust us) that the solution to the above equation is

$$\phi = e^{-ik_\mu x^\mu} \tag{4.25}$$

In order to verify this solution let's look at the first term of (4.24).

$$\begin{aligned}
\Box\phi &= \partial_\mu \partial^\mu e^{-ik_\alpha x^\alpha} \\
&= \partial_\mu \eta^{\mu\nu} \partial_\nu e^{-ik_\alpha x^\alpha} \\
&= \eta^{\mu\nu} \partial_\mu (-ik_\nu e^{-ik_\alpha x^\alpha}) \\
&= \eta^{\mu\nu} (-ik_\mu)(-ik_\nu) e^{-ik_\alpha x^\alpha} \\
&= -k_\mu k^\mu e^{-ik_\alpha x^\alpha} \\
&= -k_\mu k^\mu \phi
\end{aligned} \tag{4.26}$$

Plugging this result back into (2.17) gives that

$$(-k_\mu k^\mu + m^2)\phi = 0 \tag{4.27}$$

We showed earlier in the section that for a massive free particle, it will always hold that $k_\mu k^\mu = m^2$, and it holds here as well. This proves that the Lagrangian which we claimed to describe a free particle actually does. Furthermore, we know that if we ever encounter these terms in any Lagrangian, they will always represent the presence of a massive particle if the second term in the Lagrangian is present and represent a massless free particle if the second term is not present.

## 4.3 E+M Field

Maxwell's Equations describing electromagnetism can be written in terms of a Lagrangian just as Newton's Laws can. The lagrangian for Maxwell's theory is

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \tag{4.28}$$

Where $j^\mu = (\rho, \vec{J})$, and $A^\mu = (V, \vec{A})$ is the 4-vector potential field. For a free theory, we see that $j^\mu = 0$, and so we will consider the lagrangian for Maxwell's theory to be

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{4.29}$$

The term $F_\mu \nu$ is a rank two antisymmetric tensor called the strength tensor. It is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.30}$$

and consists of terms which describe the electric and magnetic fields of the theory. We can write the strength tensor as

$$T^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{4.31}$$

Now in order to determine the equations of motion we can follow the same method as in the previous sections. Let's first write out the lagrangian in a way which is easier to work with. We can write

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= 2(\partial_\mu A_\nu)(d^\mu A^\nu) - 2(\partial_\mu A_\nu)(d^\nu A^\mu) \end{aligned} \tag{4.32}$$

So then we can see

$$\begin{aligned} \mathcal{L} &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= -\tfrac{1}{2}(\partial_\mu A_\nu)(d^\mu A^\nu) + \tfrac{1}{2}(\partial_\mu A_\nu)(d^\nu A^\mu) \end{aligned} \tag{4.33}$$

Now if we vary the Lagrangain with respect to the vector potential so that $A_\mu \to \delta A_\mu$ we can write that

$$\delta\mathcal{L} = -(\partial_\mu \delta A_\nu)(\partial^\mu A^\nu) + (\partial_\mu \delta A_\nu)(\partial^\nu A^\mu) \tag{4.34}$$

We can take this term and integrate it by parts to get that

$$\begin{aligned} \delta\mathcal{L} = (\partial_\nu \partial^\nu A^\mu)\delta A_\mu &- (\partial_\nu \partial^\mu A^\nu)\delta A_\mu \\ (\Box A^\mu - \partial^\mu \partial_\nu A^\nu)\delta A_\mu &= 0 \\ \Box A_\mu - \partial_\mu \partial_\nu A^\nu &= 0 \end{aligned} \tag{4.35}$$

This equation represents the set of four Maxwell Equations. We can notice immediately that unlike the case of the classical scalar field, we do not have a mass term present. So therefore we can interpret the particles described by this theory as massless photons. Now, we can also note that the solution to equation above is a wave. We would thus expect that there would be two physical solutions, representing the two polarizations that the photons can have. However, $A_\mu$ clearly has four degrees of freedom. So what are the two extra solutions which are appearing?

First we can look at the last line of equation (4.35) with $\mu = 0$ and see that we would get

$$\begin{aligned} \Box A_0 - \partial_0 \partial_\nu A^\nu &= 0 \\ (\partial_0 \partial^0 + \partial_j \partial^j)A_0 - \partial_0 \partial^0 A_0 - \partial_0 \partial^j A_j &= 0 \\ \partial_j \partial^j A_0 - \partial_0 \partial^j A_j &= 0 \end{aligned} \tag{4.36}$$

We see that there is no $\Box$ term for $A_0$ and thus we can realize that $A_0$ is not a physically propagating field. It is instead known as an auxiliary field, which has no physical meaning.

This, however, still leaves us with three degrees of freedom when we only expect to have two.

It turns out that the E+M theory has a gauge invariance, and so we end up with one redundant gauge degree of freedom. We can show that the theory has this gauge invariance, and also that if we pick a specific gauge, this last gauge degree of freedom goes away. Remember from the introduction that gauge invariance comes from the existence of a mathematical transformation which can be performed on the theory, leaving it unchanged; this is what is called a gauge transformation. For our E+M theory, the gauge transformation is a transformation of the potential field $A_\mu$. Specifically, the gauge transformation is when we let:

$$A_\mu \to A_\mu + \partial_\mu \Lambda \tag{4.37}$$

where $\Lambda$ is any arbitrary scalar field which changes $A_\mu$ infinitesimally. This term can be referred to as the gauge parameter since we are free to choose it. We can see explicitly that this transformation leaves the theory unchanged.

$$\begin{aligned} F_{\mu\nu} \to F'_{\mu\nu} &= \partial_\mu(A_\nu + \partial_\nu\Lambda) - \partial_\nu(A_\mu + \partial_\mu\Lambda) \\ &= \partial_\mu A_\nu + \partial_\mu\partial_\nu\Lambda - \partial_\nu A_\mu - \partial_\nu\partial_\mu\Lambda \\ &= F_{\mu\nu} \end{aligned} \tag{4.38}$$

The strength tensor with upper indices transforms in the same way as above. Thus by (4.30) we can see that the E+M theory is invariant under this transformation. Now let's go ahead a pick a specific gauge parameter since we have shown that we are free to do so. Let's first look at the transformation on $A_\mu$.

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda \tag{4.39}$$

Then taking the derivative of this would give

$$\begin{aligned} \partial^\mu A'_\mu &= \partial^\mu A_\mu + \partial^\mu \partial_\mu \Lambda \\ &= \partial^\mu A_\mu + \Box\Lambda \end{aligned} \tag{4.40}$$

So we can see that a logical choice for $\Lambda$ would be if we let

$$\Box\Lambda = -\partial^\mu A_\mu \tag{4.41}$$

Thus, we can say that we have fixed the gauge to be

$$\partial_\mu A^\mu = 0 \tag{4.42}$$

This choice of gauge is what is known as the Lorentz gauge. So how does this invariance help to eliminate the last degree of freedom? Remember that the strength tensor is just an object used to describe the E+M field. It is also a function of the potential field $A_\mu$. We are free to choose any gauge parameter in defining this potential field, and thus by choosing a specific value for this gauge parameter, we loose a degree of freedom. If we look back at (4.35) and substitute in out fixed gauge, we can see that we are left with the equation

$$\Box A_\mu = 0 \tag{4.43}$$

to describe the E+M field with the two appropriate degrees of freedom.

In summary, we first wrote out the Lagrangian for Maxwell's theory. We found the equations of motion, but saw that there would be four solutions. The field has no massive mode, so it must describe photons, which we know should have only two modes, representing their two possible polarizations. We saw immediately that one of these degrees of freedom was an auxiliary field, which would not propagate and thus has no physical meaning. We were able to show that the last unwanted degree of freedom came from the gauge invariance of the theory. By picking a gauge (one which conveniently simplified our solution) we were able to get rid of this last degree of freedom, leaving us with only 2 physically propagating fields, representing the two transverse modes of the massless photon. We will see that this same type of invariance comes up when looking at a field theory of GR.

## 4.4 GR Free Space

Just as we did in the previous sections, we will start here with the Lagrangian for Einstein's GR. By performing field variations we will be able to reproduce Einstein's equations. We will also find that there is a gauge symmetry in Einstein's theory, which can again be used to eliminate redundant degrees of freedom from the solutions to equations of motions. Let's start by stating the Lagrangian

$$\mathcal{L} = \sqrt{-g}R \tag{4.44}$$

While this looks exceedingly simple, it is shorthand for a longer expression. Here, $g = |g_{\mu\nu}|$ where $g_{\mu\nu}$ is the metric. The metric tensor is a rank two tensor which describes the curvature of an associated four-dimensional spacetime. R is known as the Ricci tensor, which is the contraction of $R_{\mu\nu}$, so that:

$$R = R^\mu_\mu = g^{\mu\nu} R_{\mu\nu} \tag{4.45}$$

In turn, $R_{\mu\nu}$ is the contraction of $R^\lambda_{\mu\lambda\nu}$, the Riemann curvature tensor, which can be written in full as:

$$R^\lambda_{\mu\lambda\nu} = \partial_\mu \Gamma^\rho_{\mu\sigma} - \partial_\nu \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{4.46}$$

Finally, $\Gamma$ is the Christoffel symbol, which is a function of the metric tensor and takes the form:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\mu\nu}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \tag{4.47}$$

Now, in order to find the equations of motion, we must vary the Lagrangian with respect to the metric tensor. We should note first, however, that we have a choice here. The metric tensor comes in two forms, one with upper indices and one with lower indices. Both represent the same information and both can be written in terms of the other, so we are free to choose which one we will vary the Lagrangian with. We will choose to vary with respect to the version with upper indices, $g^{\mu\nu}$, which is a convention taken by many. Variation of the metric $g_{\mu\nu}$ with respect to $g^{\mu\nu}$ can be found as follows. We have that

$$g^{\mu\nu}g_{\mu\nu} = \delta^{\nu}_{\sigma} \tag{4.48}$$

where $\delta^{\nu}_{\sigma}$ is constant. Thus

$$\begin{aligned} (\delta g^{\mu\nu})g_{\nu\sigma} + g^{\mu\nu}(\delta g_{\nu\sigma}) &= 0 \\ \delta g^{\mu\nu}g_{\nu\sigma} &= -g^{\mu\nu}\delta g_{\nu\sigma} \\ \delta g^{\mu\nu}g^{\sigma\rho}g_{\nu\sigma} &= -g^{\mu\nu}g^{\sigma\rho}\delta g_{\nu\sigma} \\ \delta g^{\mu\nu}\delta^{\rho}_{\nu} &= -g^{\mu\nu}g^{\sigma\rho}\delta g_{\nu\sigma} \\ \delta g^{\mu\rho} &= -g^{\mu\nu}g^{\sigma\rho}\delta g_{\nu\sigma} \end{aligned} \tag{4.49}$$

And finally we see that

$$\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma} \tag{4.50}$$

Now we can vary the Lagrangian in (2.44). Remembering that $R = g^{\mu\nu}R_{\mu\nu}$ and applying the chain rule, we get that

$$\delta\mathcal{L} = \delta(\sqrt{-g})g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}(\delta g^{\mu\nu})R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}(\delta R_{\mu\nu}) \tag{4.51}$$

The second term in the expression above is fine as it is. In the first term, we need to find $\delta\sqrt{-g}$, and in the last term we need to find $\delta R_{\mu\nu}$, both in terms of $\delta g^{\mu\nu}$. Let's do the first of these. Using the identity that for a matrix M (in this case, $g^{\mu\nu}$)

$$ln(detM) = Tr(lnM) \tag{4.52}$$

we can find that

$$\begin{aligned} \delta g &= g g^{\mu\nu}\delta g_{\mu\nu} \\ &= -g g_{\mu\nu}\delta g^{\mu\nu} \end{aligned} \tag{4.53}$$

Then we can take

$$\delta\sqrt{-g} = \tfrac{1}{2}\tfrac{1}{\sqrt{-g}}(-\delta g) \tag{4.54}$$

and substitute in our expression for $\delta g$ to get

$$\begin{aligned} \delta\sqrt{-g} &= \tfrac{1}{2}\tfrac{1}{\sqrt{-g}}(g g_{\mu\nu}\delta g^{\mu\nu}) \\ &= -\tfrac{1}{2}\tfrac{-g}{\sqrt{-g}}g_{\mu\nu}\delta g^{\mu\nu} \\ &= -\tfrac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \end{aligned} \tag{4.55}$$

Which completes the task of finding $\delta g$ in terms of $\delta g^{\mu\nu}$. Next we can look at the $\delta R_{\mu\nu}$ term. Using identities for $\Gamma$, it can be shown that this term goes to zero (see Carroll pg. 162). So we get that

$$\delta R_{\mu\nu} = 0 \tag{4.56}$$

Substituting (4.55) and (4.56) back into equation (4.51), we see that

$$\begin{aligned}\delta\mathcal{L} &= -\tfrac{1}{2}\sqrt{-g}g_{\mu\nu}(\delta g^{\mu\nu})g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}(\delta g^{\mu\nu})R_{\mu\nu} \\ &= \sqrt{-g}[R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R]\delta g^{\mu\nu}\end{aligned} \tag{4.57}$$

We can define the Einstein tensor, labelled as $G_{\mu\nu}$ to be

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R \tag{4.58}$$

We know that varying the Lagrangian must yield zero, so the equation(2.66) must be zero, and thus our newly defined Einstein tensor must be zero. Letting

$$G_{\mu\nu} = 0 \tag{4.59}$$

gives a set of 10 equations, which are the Einstein equations in free space.

Just as Maxwell's theory described massless photons, this theory describes massless particles which would be referred to as gravitons, the quantum unit for gravity. In order to look at these more closely, we can think of the metric for a curved spacetime as just the metric for a flat spacetime ($\eta_{\mu\nu}$)plus a small perturbation, which we'll call $h_{\mu\nu}$. We can rewrite the Einstein tensor in terms of this small perturbation. Skipping all of the messy step, we would get (see Carroll for a full description)

$$G_{\mu\nu} = 0 = \tfrac{1}{2}(\partial_\sigma\partial_\nu h^\sigma_\mu + \partial_\sigma\partial_\mu h^\sigma_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu}\Box h) \tag{4.60}$$

Here, $h$ is the contraction $h = \eta^{\mu\nu}h_{\mu\nu} = h^\mu_\mu$. We will now go on to show that this equation really only has two independent and meaningful solutions. This may at first seem surprising, since upon first glance we would except there to be 16 solutions (because $h_{\mu\nu}$ is a rank two tensor with 16 components.) However, we know that $h_{\mu\nu}$ is symmetric, and so only has 10 independent components. This brings our total number of degrees of freedom down to 10. Next, we can note a gauge symmetry of the theory. It turns out that the relevant gauge transformation here is

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu \tag{4.61}$$

If we were to plug this back into equation (4.60), we would find that all of the $\xi_\mu$ terms cancel and that we would get the same equation back; it would leave the theory invariant. Now we can define the term

$$\partial_\mu h^{\bar{\mu\nu}} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h \tag{4.62}$$

And again choose the Lorentz gauge on the perturbation field, writing that

$$\partial_\mu \bar{h^{\mu\nu}} = 0 \tag{4.63}$$

This is really a set of four different equations, so where we lost one degree of freedom in the E+M theory, we now lose four degrees of freedom. This reduces our total number of degrees of freedom from 10 to 6. Finally, just as we had an auxiliary field in the E+M theory, we also have it here. When we have $h_{\mu\nu} = h_{0\mu}$, we see that the perturbation has no $\Box$ term with it. Thus, the four perturbation fields represented by $h^{00}$ and $h^{0j}$ are non-physical, and thus can be thrown out, thereby reducing the total number of physical degrees of freedom to two. If we rewrite the Einstein tensor with our specific choice of gauge, we see that equation (4.60) becomes

$$\Box \bar{h}_{\mu\nu} = 0 \tag{4.64}$$

This equation represents Einstein's equations.

Let's quickly review what we have done here. We started with a Lagrangian for Einstein's GR. We varied this Lagrangian with respect to the metric (we chose to vary with respect to the metric with upper indices) in an effort to find the equations for motion. The equations of motion, Einstein's equations, were found. Furthermore, we saw that we could represent the curved spacetime defined by the metric $g_{\mu\nu}$ as flat Minkowski space, plus a small perturbation. We found that Einstein's equations initially appeared to have a total of 16 solutions, one for each degree of freedom. However, we were able to get rid of many of these solutions. Six were thrown out since the perturbation on Minkowski space is symmetric. Another four were able to be gauged away. Finally, another set of four were seen to be non-propagating and thus non-physical fields. So, we are left with $(16 - 6 - 4 - 4) = 2$ degrees of freedom, and thus two physical solutions. This represents the two transverse modes (polarizations) of a massless graviton.

# 5 Symmetries in Curved Spaces

## 5.1 Manifolds

As we have said before, studying GR involves learning a lot of mathematics. General relativity involves working with curved spaces. These curved spaces are treated mathematically as manifolds, which are just generalizations of our usual flat Euclidean spaces. While this may be hard to visualize, we'll be able to see that when you zoom into a small enough region of a curved manifold, it looks essentially like flat space. The manifold can thus be created by piecing many of these very small regions together smoothly.

So the idea of the manifold is really quite straightforward. Let's now go about expressing the idea of the manifold in a mathematically rigorous way. First we should talk about mappings between manifolds. Let's consider the idea of a function. A function is just a specific type of mapping, between two $R^n$ dimensional spaces. A mapping is just a generalization of this concept; a map $\phi : M \to N$ assigns an element of a manifold M to an element of another manifold N.

Next, we also have to rethink the concept of vectors and tensors when we're working in the setting of a manifold. Remember, in a Euclidean space, a vector is an object which points from one location to another. When we're working in a curved space, however, this definition no longer holds. Instead a vector can be thought of as an operator which acts of the manifold. The vector must be defined at some point, p, in the manifold. It then operates on the space of all smooth functions at the same point p. The vector exists in a space tangent to the manifold, which is defined by the space of directional derivative operators along curves through p, as shown below in Figure 1.

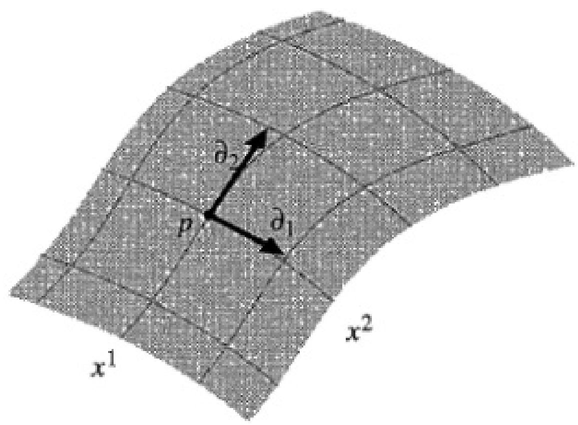


Figure 1: A manifold's tangent space, defined by the space of directional derivative operators along curves through p. Figure recreated from Ref. [1].

We can also define the dual vector, which also is an object that exists at some point, p, and that operates on the manifold. The dual vector exists in a cotangent space associated with the manifold at the point p. We have a tangent space,$T_p$ for the manifold at a point p which has a basis formed by the partial derivative operators $\partial_\mu$ at p (where we have imposed a coordinate chart with coordinates $x^\mu$.) The cotangent space $T_p^*$ is then defined as the set of linear maps $\omega : T_p \to \mathbb{R}$.

Now that we have defined the vector and the dual vector in the context of manifolds, we can expand the idea of the tensor into a manifold. Just as in flat space, a (k,l) tensor it is just a multilinear map from $k$ dual vectors and $l$ vectors to $\mathbb{R}$. Thus, the dual vectors are the contravariant components of the tensor and the vectors are the covariant components of the tensor. Tensors obey the property that they transform properly under general coordinate transformations. Thus, given a tensor $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$, and transforming from coordinates $x^\mu$ to $x^{\mu'}$ would give the tensor in the new coordinate system to be:

$$T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_l'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \cdots \frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \tag{5.1}$$

### 5.2 Pullback and Pushforward

Now that we have seen how vectors and tensors generalize to manifolds, we can further our understanding of these abstract spaces by looking more closely at how we can map between two manifolds. Let's consider two manifolds M and N, with M having the coordinates $x^\mu$ and N having the coordinates $y^\alpha$ (thus the manifolds need not have the same dimensionality.) Now consider again that we have a map $\phi : M \to N$, and another map (which in this case will be a function) $f : N \to \mathbb{R}$. What we would like to be able to do is move directly between M and $\mathbb{R}$. We can do this by composing the maps $\phi$ and $f$ to create a map $(f \circ \phi) : M \to \mathbb{R}$. We call this composition the pullback of $f$ by $\phi$ which gets denoted by $\phi^* f$.

$$\phi^* f = (f \circ \phi) \tag{5.2}$$

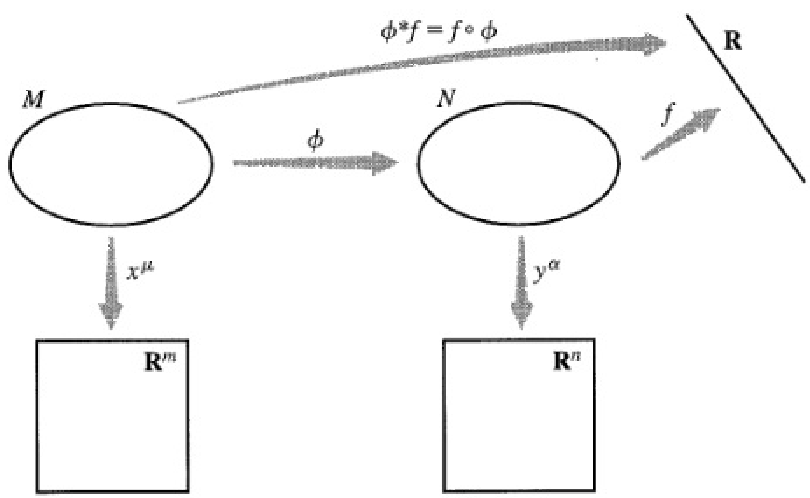


Figure 2: Chart diagramming the relations between manifolds $M$ and $N$ when a function $f$ is pulled back by $\phi$. Figure recreated from Ref. [1].

This process makes perfect sense and is quite useful when moving between two manifolds. Now, what if we want to do the opposite? That it to say, we have a function $g : M \to \mathbb{R}$ which we want to combine with our map $\phi : M \to N$ to move from N to $\mathbb{R}$. There is no way for us to simply compose the function $g$ with the map $\phi$ in order to create a new mapping. How do we go about fixing this problem, since surely there must be a way to use the information which we have to get the desired effect.

Remember that a vector on a manifold can be thought of as a derivative operator that maps smooth functions to real numbers. We can use this property to define the pushforward of a vector. If we have a vector on the manifold M at point p, V(p), then the pushforward vector $\phi_* V$ at point $\phi(p)$ on N is given by its action on functions on N, which we'll denote $\mathcal{F}(N)$, so that

$$(\phi_* V)(\mathcal{F}(N)) = V(\phi^* \mathcal{F}(N)) \tag{5.3}$$

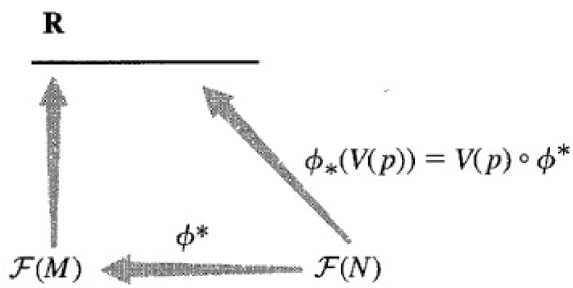


Figure 3: A symbolic diagram of the process of pushing forward a vector. Figure recreated from Ref. [1].

We can now see that it is possible to push some objects forward and pull other objects back. While we will not show it in this thesis, it can be proven that vectors can be pushed forward but not pulled back, and dual vectors can be pulled back but not pushed forward. Now, since tensors are just maps from a set of vectors and dual vectors to $\mathbb{R}$, we can figure out which tensors are allowed to be pushed forward and which can be pulled back. A (0,j) tensor $T_{\alpha_1 \cdots \alpha_j}$ can only be pulled back (since it involves only dual vectors.) The pullback of T is

$$(\phi^* T)_{\mu_1 \cdots \mu_j} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_j}}{\partial x^{\mu_j}} T_{\alpha_1 \cdots \alpha_j} \tag{5.4}$$

Likewise, a (k,0) tensor $S_{\mu_1 \cdots \mu_k}$ can only be pushed forward (since it involves only vectors.) The pushforward of S is:

$$(\phi_* S)^{\alpha_1 \cdots \alpha_k} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_k}}{\partial x^{\mu_k}} S^{\mu_1 \cdots \mu_k} \tag{5.5}$$

It should also be noted that tensor which have both upper indices and lower indices (and thus involves both vectors and dual vectors) are generally not allowed to be either pushed forward or pulled back. This can only occur if we map a manifold $M$ back to itself, thereby using an invertible map.

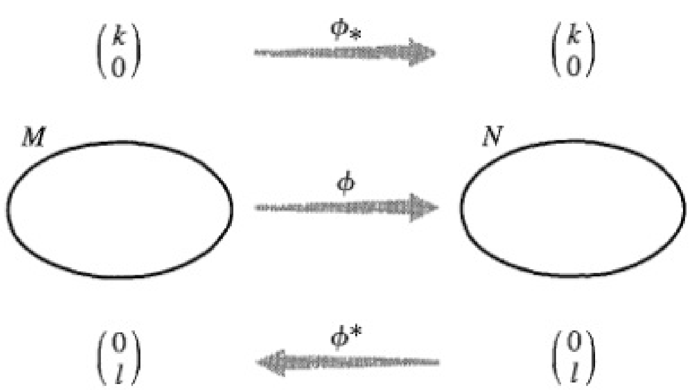


Figure 4: A chart showing the (generally) allowed manipulation on tensors. Figure recreated from Ref. [1].

Now that we are starting to get a better understanding of the mathematics of vectors and tensors on manifolds, we will go on to look at diffeomorphisms.

### 5.3 Diffeomorphisms

We saw in the last section that (k,0) tensors can be pushed forward but not pulled back and (0,j) can be pulled back but not pushed forward. Why is this? It stems from the fact that the map $\phi : M \to N$ is, in general, not invertible. So now, what happens if we can be assured that the map $\phi$ is invertible? If this is true (and, as an addendum, both the map $\phi$ and the inverse map $\phi^{-1}$ are $C^\infty$ mappings) then the map is said to be a diffeomorphism.

In the theory of General Relativity, we use the concept of a manifold to represent a curved spacetime. In this context, we are only ever concerned with mappings from a manifold back onto itself. For example, if the manifold M represents the spacetime around a massive black hole, we only care about mappings which take us from one location on M to another location on M. If we call this map $\phi : M \to M$, then clearly it will invertible since it is mapping back onto itself. As long as the map is smooth (infinitely differentiable) the map $\phi$ will be a diffeomorphism. This mapping $\phi$ is especially useful because we are now allowed to push forward or pull back any tensor, vector, or scalar. This property allows us to investigate how objects, especially tensors, change under a diffeomorphism. This will be done by looking at what is known as a Lie derivative, which is a generalization of our familiar directional derivative.

Before we go on to look at the Lie derivative, let's first take a step back. Everything that we have done in this section is extremely mathematical, and it is easy to get lost in the terminology. The following is a quote from Carlo Rovelli's book *Quantum Gravity* (see Ref.[3]). It is a good conceptual example of a diffeomorphism on the surface of the Earth (which can be seen, of course, as a two-dimensional manifold.)

> "Consider the surface of the Earth as a manifold, and call it M. At each point P $\in$ M on Earth, say the city of Paris, there is a certain temperature T(P). The temperature is a scalar function $T : M \to \mathbb{R}$ on the Earth's surface. Imagine a simplified model of weather evolution in which the only factor determining temperature change was the displacement of air due to wind. By this I mean the following. Fix a time interval: say we call T the temperature on May $1^{st}$, and $\tilde{T}$ the temperature on May $2^{nd}$. During this time interval, the winds move the air which is over a point $Q = \phi(P)$ to point P. If, say, Q is the French village of Quintin, this means that the winds have blown the air of Quintin to Paris. Assume the temperature $\tilde{T}$(P) of Paris on May $2^{nd}$ is equal to the temperature T(Q) of Quintin the day before. The "wind" map $\phi$ is a map from the Earth's surface to itself, which associates with each point P the point Q from which the air has been blown by the wind. Assuming it is smooth and invertible, the map $\phi : M \to M$ is an active diffeomorphism."

### 5.4 Lie Derivative

In a flat space, the concept of a derivative is relatively straightforward; it is just the rate of change of one variable with respect to another variable. However, when we're working in the context of manifolds, this becomes much more difficult to see. Lets use the following example to show this. Again consider a sphere as two-dimensional manifold. At point P exists a vector with some specific orientation in the tangent space to the sphere at point P. This vector is then moved along some closed path $\gamma$ (which, to make this example work, is not a great circle on the sphere) so that it ends up back at point P. This vector will *not* be pointing in the same direction as before it was moved. This is true too for tensors. If a tensor is moved from some point P to some other point Q (as the result of a diffeomorphism, for example) the tensor must be pulled back to its original location P and have it's new orientation compared to its original location in order to account for the differences due to the curvature of the manifold. This is illustrated in the figure below. For tensors the situation is even slightly more complicated because they involve both vectors and dual vectors.

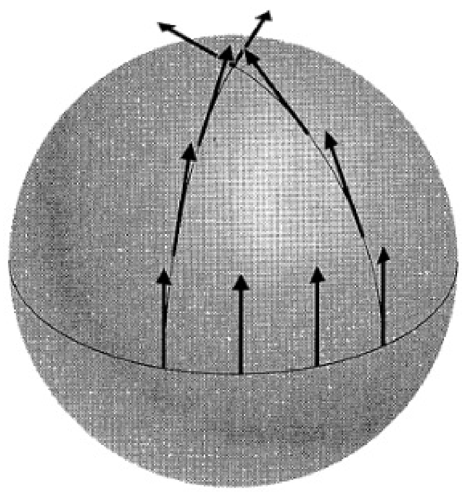

Figure 5: The act of moving a vector along a path on a curved space (in this case a sphere.) Figure recreated from Ref. [1].

The Lie derivative is used to effectively measure changes in some quantity on a manifold. Let's first consider a diffeomorphism which takes us from the coordinate $x^\mu$ to the coordinate $X^\mu + V^\mu$, where $V^\mu$ is some infinitesimal change on the manifold. With this in mind, the change in a tensor, vector, or scalar U on the manifold is given by Lie derivative, with th notation $\mathcal{L}_V U$.

For a scalar field $\phi$, the Lie derivative is given by the ordinary directional derivative:

$$\mathcal{L}_V \phi = V(\phi) = V^\mu \partial_\mu \phi \tag{5.6}$$

Now let's extend the Lie derivative for a vector field. First, we'll choose to work in a coordinate system $x^\mu = (x^1, \cdots, x^n)$ such that $x^1$ is the parameter along the integral curves.

Then we'll look at a vector field of the form $V = \frac{\partial}{\partial x^1}$, which, in the coordinate system we've chosen, has components $V^\mu = (1, 0, \cdots, 0)$. If we then perform a diffeomorphism by amount t on the vector field, it just amounts to a coordinate transformation from $x^\mu$ to $y^\mu = (x^1 + t, x^2, \cdots, x^n)$. From this, we can say that the components of the tensor pull back from point $\phi_t(p)$ to point $p$ are:

$$\phi_t^*[T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}(\phi_t(p))] = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}(x^1 + t, x^2, \cdots, x^n) \tag{5.7}$$

The Lie derivative would then be given by:

$$\mathcal{L}_V T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = \frac{\partial}{\partial x^1} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \tag{5.8}$$

If this tensor field is a vector field $U^\mu(x)$ then the Lie derivative is:

$$\mathcal{L}_V U^\mu = \frac{\partial U^\mu}{\partial x^1} \tag{5.9}$$

In the coordinate system, we can see that the commutator $[V, U]^\mu$ is:

$$\begin{aligned} [V, U]^\mu &= V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu \\ &= \frac{\partial U^\mu}{\partial x^1} \end{aligned} \tag{5.10}$$

We can see that the in this coordinate system the Lie derivative of U with respect to V has the same components as the commutator of V and U. However, because both V and U are vector fields, it must be true that they are the same in any coordinate system. Thus, the commutator of V and U is referred to as the Lie bracket, and the Lie derivative for a vector field is defined as:

$$\mathcal{L}_V U^\mu = [V, U]^\mu \tag{5.11}$$

The Lie derivative for a tensor field can be found using similar ideas. For a tensor with an arbitrary number of upper and lower indices, the Lie derivative takes the form:

$$\begin{aligned} \mathcal{L}_V T^{\mu_1 \mu_2 \cdots \mu_k}{}_{\nu_1 \nu_2 \cdots \nu_l} &= V^\sigma \partial_\sigma T^{\mu_1 \mu_2 \cdots \mu_k}{}_{\nu_1 \nu_2 \cdots \nu_l} \\ &- (\partial_\lambda V^{\mu_1}) T^{\lambda \mu_2 \cdots \mu_k}{}_{\nu_1 \nu_2 \cdots \nu_l} \\ &- (\partial_\lambda V^{\mu_2}) T^{\mu_1 \lambda \cdots \mu_k}{}_{\nu_1 \nu_2 \cdots \nu_l} - \cdots \\ &+ (\partial_{\nu_1} V^\lambda) T^{\mu_1 \mu_2 \cdots \mu_k}{}_{\lambda \nu_2 \cdots \nu_l} \\ &+ (\partial_{\nu_2} V^\lambda) T^{\mu_1 \mu_2 \cdots \mu_k}{}_{\nu_1 \lambda \cdots \nu_l} + \cdots \end{aligned} \tag{5.12}$$

For a more complete discussion of the Lie derivative, see Ref. [1] and Ref. [2].

## 5.5 Symmetries, Isometries, and Killing Vectors

When we say that a manifold possesses a symmetry, we mean that the geometry is invariant under some transformation that maps M to itself. This is, however, a very broad definition for what we mean when we say that a symmetry exists. Luckily, we have developed most of the machinery necessary to create a more useful definition for a symmetry.

We said that a symmetry involves some transformation that maps $M$ back to itself. This transformation is precisely the diffeomorphism that we have examined. We say that a diffeomorphism $\phi$ is a symmetry of a tensor $T$ if the tensor is invariant after it has been pulled back under $\phi$. That is to say, $\phi$ is a symmetry of $T$ if

$$\phi^* T = T \tag{5.13}$$

This matches well with our more general definition. Under some transformation, now identified as a diffeomorphism, the manifold is essentially the same from point to point. Now we can see that it is possible for there to be multiple diffeomorphisms which allow equation (3.13) to be true. In this case, we would have a one-parameter family of symmetries, which we can denote by $\phi_t$. This family of symmetries forms a vector field $V^\mu(x)$. So, it must be true that the Lie derivative of the tensor T with respect to the vector field $V^\mu(x)$ is equal to zero

$$\mathcal{L}_V T = 0 \tag{5.14}$$

When studying GR, the most important types of symmetries are those of the metric. We saw in section 2.4 that the metric, denoted $g_{\mu\nu}$, is a rank-two tensor which describes the curvature of a four-dimensional spacetime. Having a symmetry of the metric means that the curvature of the space is the same under some transformation. This type of symmetry is so important that it gets its own name. A diffeomorphism $\phi$ is called an *isometry* if:

$$\phi^* g_{\mu\nu} = g_{\mu\nu} \tag{5.15}$$

As we said before, $\phi$ can be a one-parameter family of diffeomorphisms, or in this case, a one-parameter family of isometries. This family of isometries creates a vector field, which is now referred to as a Killing vector field, and denoted $K^\mu(x)$. So, we can rewrite equation (3.14) with our specific tensor and vector field to see that:

$$\mathcal{L}_K g_{\mu\nu} = 0 \tag{5.16}$$

Now, using equation (3.12), we can write out this equation in full.

$$\begin{aligned} \mathcal{L}_K g_{\mu\nu} &= K^\sigma \nabla_\sigma g_{\mu\nu} + (\nabla_\mu K^\lambda) g_{\lambda\nu} + (\nabla_\nu K^\lambda) g_{\mu\lambda} \\ &= \nabla_\mu K_\nu + \nabla_\nu K_\mu \\ &= 2\nabla(_\mu K_\nu) \end{aligned} \tag{5.17}$$

We have replaced the partial derivatives from (3.12) with the del operator, and we'll leave it to the reader to verify that the equation written in this way is equivalent (all extra terms will simply cancel.)

Combining equations (3.17) and (3.16) shows that for a Killing vector field $K^\mu(x)$:

$$\nabla_{(\mu} K_{\nu)} = 0 \tag{5.18}$$

Equation (3.18) is known as Killing's equation. If a vector satisfies this equation, then there must exist a coordinate system in which the metric is independent of one of its coordinates. Thus, for every Killing vector that exists there is a corresponding symmetry of the metric. The existence of a Killing vector means that there is an isometry. By counting the number of killing vectors, we can find out how many symmetries exists for a given metric.

# 6 Spontaneous Symmetry Breaking

Let's imagine that a group of physicists have all been invited to a dinner party. Everyone present sits down at a circular table, each with a place setting and a glass of water evenly spaced between each person. The physicists, not realizing that their glass is always placed to their right, are free to choose either glass available. However, assuming that each person at the table must have a glass, the moment at which one of the party guests decides which side to take the glass from it has been decided which side all other guests will take their glass from. Each guest initially had a set of two possible solutions to the problem of which glass to drink from, each equally valid, but when the choice was actually made, the symmetry of the system was lost. This phenomenon is known a spontaneous symmetry breaking.

We have seen in the previous section of this thesis that theories contain symmetries when certain transformations exist which leave the theory invariant. We went through in a fair amount of detail and showed that a theory can have many different solutions, all of which are equally valid. In particle physics spontaneous symmetry breaking occurs when a symmetry, which holds dynamically for the theory, is lost for the ground-state solution, also known as the vacuum solution.

In this section we will show a few explicit examples of spontaneous symmetry break. First we'll consider a simple scalar field theory. Then, we'll go on to look at a global U(1) (unitary) symmetry, and then at local U(1) symmetry. A global symmetry is a type of symmetry which is the result of invariance to a global transformation, which is to say a transformation which is the same throughout a manifold. Likewise, a local symmetry is a symmetry which is the result of a local transformation, or a transformation which is definitely uniquely at every point in the manifold.

## 6.1 Scalar Field Theory

Let's start by considering a simple Lagrangian consisting of a kinetic and a potential term.

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) \tag{6.1}$$

where the potential term $V(\phi)$ takes the form

$$V(\phi) = \tfrac{1}{2} m^2 \phi^2 + \tfrac{1}{4}\lambda \phi^4 \tag{6.2}$$

where $\lambda > 0$. We can see immediately that the Lagrangian is invariant under the parity transformation of the scalar field $\phi$ such that

$$\phi \to -\phi \tag{6.3}$$

First, let's consider the case where $m^2 > 0$. Since $\lambda$ is a positive constant, we know that $V$ has a unique minimum at $\phi = 0$. This is shown in the figure below.

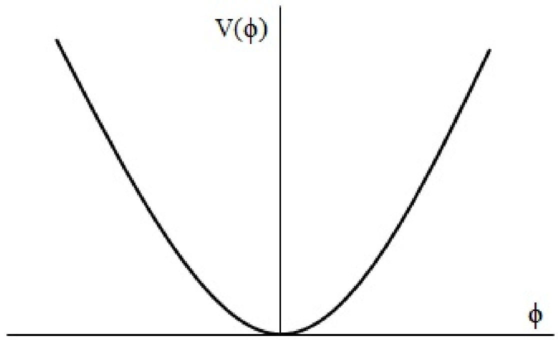


Figure 6: The potential function of equation (6.2) with $m^2 > 0$. This function has a unique minimum.

We write this minimum value as $\langle\phi\rangle = 0$, where the quantity $\langle\phi\rangle$ is referred to as the vacuum expectation value. The vacuum expectation value, or vev, gives the ground state of the system.

Now, let's consider small oscillations about this vev, which we'll label as $\epsilon$. We can then say that in the case of the small oscillation, we have

$$\begin{aligned}\phi &= \langle\phi\rangle + \epsilon \\ &= 0 + \epsilon \\ &= \epsilon\end{aligned} \tag{6.4}$$

If we substitute this back into equations (6.1) and (6.2), and keep only the second order terms, we get that

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \epsilon)(\partial^\mu \epsilon) - \tfrac{1}{2}m^2\epsilon^2 + O(\epsilon^2) \tag{6.5}$$

Since $\epsilon$ is already small, higher powers of it will be unimportant. If we refer back to our discussion of classical field theory, we can recognize this Lagrangian as describing a free massive particle $\epsilon$. Clearly no symmetries have been broken.

If we want to look at spontaneous symmetry breaking, we can study the case where we let $m^2 < 0$, which still has symmetry under $\phi \to -\phi$. The potential term now has a minimum at

$$\tfrac{\partial V}{\partial \phi} = m^2\phi + \lambda\phi^3 = 0 \tag{6.6}$$

Which yields that the vev is

$$\langle\phi\rangle = \pm\sqrt{\tfrac{-m^2}{\lambda}} \tag{6.7}$$

The potential function is shown in the figure below, where the two minimums have the values in equation (6.7).

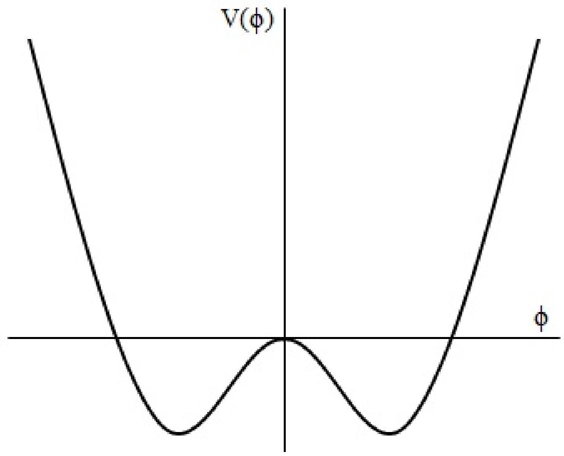


Figure 7: The potential function of equation (6.2) with $m^2 < 0$. This function has a set of 2 minima.

Clearly, there is no reason as to whether we should choose the positive or negative sign in this case. Thus we are free to choose whichever one we prefer. Let's pick the positive sign and define it as $v$, such that

$$\langle\phi\rangle = \pm\sqrt{\tfrac{-m^2}{\lambda}} \equiv v \tag{6.8}$$

Now we can change our coordinates to be located at our chosen vacuum value. We can let

$$\begin{aligned} \phi' &= \phi - \langle\phi\rangle \\ &= \phi - v \end{aligned} \tag{6.9}$$

Which then gives that

$$\begin{aligned} \langle\phi'\rangle &= \langle\phi\rangle - v \\ &= v - v \\ &= 0 \end{aligned} \tag{6.10}$$

Thus, we can write the Lagrangian from equations (6.1) and (6.2) in terms of $\phi'$.

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi')(\partial^\mu\phi') - (-m^2)(\tfrac{\phi'^4}{4v^2} + \tfrac{\phi'^3}{v} + \phi'^3 - \tfrac{v^2}{4}) \tag{6.11}$$

We can clearly see that we no longer have a symmetry for $\phi'$. Actually the symmetry of the system is still there, but now it is hidden. We can now again consider a small oscillation, $\epsilon$, about our chosen vacuum $\langle\phi'\rangle$, such that

$$\begin{aligned}\phi' &= \langle \phi' \rangle + \epsilon \\ &= 0 + \epsilon \\ &= \epsilon\end{aligned} \tag{6.12}$$

Again we can substitution this into our Lagrangian, equation (6.11), and, keeping only second order terms in $\epsilon$, find that

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \epsilon)(\partial^\mu \epsilon) - \tfrac{1}{2}(-2m^2)\epsilon^2 + O(\epsilon^3) \tag{6.13}$$

We can now see that this Lagrangian describes a free massive particle with a mass squared equal to $(-2m^2)$. Since we assumed that $m^2 < 0$, this would describe a particle with a positive mass, and thus described a real particle.

This is an example of spontaneous symmetry breaking In this case, since the potential could take one of two different values to be a vacuum, it is referred to as a discrete symmetry. In the next example, where we look at global U(1) symmetry, we'll see that we get sightly different results when we break a continuous symmetry.

## 6.2 Global U(1) Symmetry

First let's consider a Lagrangian for global U(1) gauge theory.

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^\dagger (\partial^\mu \phi) - V(\phi^\dagger \phi) \tag{6.14}$$

Here $\phi$ is a complex scalar field such that

$$\phi = \phi_1 + i\phi_2 \tag{6.15}$$

$$\phi^\dagger = \phi_1 - i\phi_2 \tag{6.16}$$

$$\phi^\dagger \phi = \phi_1^2 + \phi_2^2 \tag{6.17}$$

and, in analogy with the scalar field theory, the potential takes the form

$$V(\phi) = \tfrac{1}{2}m^2(\phi^\dagger\phi) + \tfrac{1}{4}\lambda(\phi^\dagger\phi)^2 \tag{6.18}$$

We can easily show that the Lagrangian is invariant under a global U(1) transformation where we let

$$\phi \to \phi' = e^{i\alpha}\phi \tag{6.19}$$

First we see that we can write the Lagrangian out as

$$\mathcal{L} = \tfrac{1}{2}|\partial_\mu \phi'|^2 - V(|\phi'|^2) \tag{6.20}$$

We can then substitute in (6.19) to get that

$$
\begin{aligned}
\mathcal{L} &= \tfrac{1}{2}|\partial_\mu e^{i\alpha}\phi|^2 - V(|e^{i\alpha}\phi|^2) \\
&= \tfrac{1}{2}(\partial_\mu e^{i\alpha}\phi)^\dagger(\partial^\mu e^{-i\alpha}\phi) - V[(e^{i\alpha}\phi)^\dagger(e^{-i\alpha}\phi)] \\
&= \tfrac{1}{2}e^{i\alpha}e^{-i\alpha}(\partial_\mu\phi)^\dagger(\partial^\mu\phi) - e^{i\alpha}e^{-i\alpha}V(\phi^\dagger\phi) \\
&= \tfrac{1}{2}(\partial_\mu\phi)^\dagger(\partial^\mu\phi) - V(\phi^\dagger\phi)
\end{aligned} \tag{6.21}
$$

And thus we see that the Lagrangian is invariant under this transformation. Now we can go ahead and look the cases of spontaneous symmetry breaking. Again we can notice that if $m^2 > 0$ we have a single vacuum expectation value. Since this case isn't very interesting, let's go on and consider the case where $m^2 < 0$. First, for ease of calculations let's let $x = \phi^\dagger\phi$. Then we can say that the minimum of the potential will occur where

$$
\begin{aligned}
\tfrac{dV(x)}{dx} = \tfrac{1}{2}m^2 + \tfrac{1}{2}\lambda x &= 0 \\
m^2 + \lambda x &= 0 \\
x &= \tfrac{-m^2}{\lambda}
\end{aligned} \tag{6.22}
$$

Since we have $x = \phi^\dagger\phi$, we see that when we solve for $\phi$ we get a whole set of solutions of the form

$$
\phi = e^{i\gamma}\sqrt{\tfrac{-m^2}{\lambda}} \tag{6.23}
$$

where $\gamma$ is some constant. This complex exponential is like the $\pm$ that we got in the case of a scalar field. Defining $\sqrt{\frac{-m^2}{\lambda}} = v$, and squaring both sides of the above equation, we can see that

$$
\phi_1^2 + \phi_2^2 = v^2 \tag{6.24}
$$

Where before we had two possible minimum values for the potential, we now have an entire set of values which lie along a circle. A plot of the potential is shown below. It is clear by looking at the plot that there is a whole set of possible minima for the potential.

Now we can introduce polar variables $\rho(x)$ and $\theta(x)$, which replace the fields $\phi_1$ and $\phi_2$. We see that with this we get $\phi = \rho e^{i\theta}$ and $\phi^\dagger = \rho e^{-i\theta}$. In this set of coordinates, we see that the space of vacua can be written as

$$
\rho = v \tag{6.25}
$$

Now we are free to choose any vacuum solution that we would like. Let's choose

$$
\begin{aligned}
\langle\rho\rangle &= v \\
\langle\theta\rangle &= 0 \\
\Rightarrow \langle\phi\rangle &= v
\end{aligned} \tag{6.26}
$$

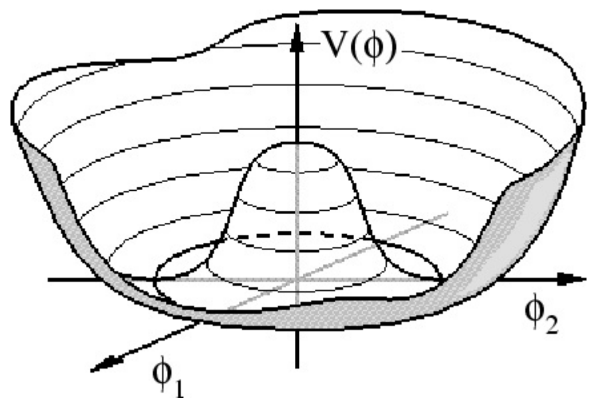


Figure 8: The potential function of equation (6.18) with $m^2 > 0$. This function is sometimes referred to as a "Mexican Hat," and has a ring of possible minima. Figure recreated from Ref. [5].

Once we've picked our vacuum solution, let's look at small excitations about this point. We'll let $h$ be an excitation up the wall of the trough of possible vacua, and $\epsilon$ be an excitation along the circle of possible vacua. We can now write

$$\begin{aligned} \rho &= v + h \\ \theta &= 0 + \tfrac{\epsilon}{v} \\ \Rightarrow \phi &= (v + h)e^{\frac{i\epsilon}{v}} \end{aligned} \tag{6.27}$$

With this we see that

$$\begin{aligned} \phi^\dagger \phi &= (v + h)e^{\frac{-i\epsilon}{v}}(v + h)e^{\frac{i\epsilon}{v}} \\ &= (v + h)^2 \end{aligned} \tag{6.28}$$

We can now substitute this into our Lagrangian. We'll first look at the potential term, then look at the kinetic term.

$$\begin{aligned} V(\phi) &= \tfrac{1}{2}m^2(\phi^\dagger\phi) + \tfrac{1}{4}\lambda(\phi^\dagger\phi)^2 \\ &= \tfrac{1}{2}m^2(v + h)^2 + \tfrac{1}{4}\lambda(v + h)^4 \\ &= \tfrac{1}{2}m^2(v^2 + 2vh + h^2) + \tfrac{1}{4}\lambda(v^4 + 2v^3h + v^2h^2 + \\ &\qquad 2v^3h + 4v^2h^2 + 2vh^3 + h^2v^2 + 2vh^3 + h^4) \end{aligned} \tag{6.29}$$

We'll get hid of all higher order $h$ terms, since it is already a small excitation. We will also ignore constant terms (terms not involving an $h$.)

$$\begin{aligned} V(\phi) &= \tfrac{1}{2}(2vh + h^2) + \tfrac{1}{4}(4v^3h + 6v^2h^2) \\ &= m^2h(v + \tfrac{h}{2}) + \lambda h(v^3 + \tfrac{3}{2}v^2h) \\ &= v(m^2 + \lambda v^2)h + (\tfrac{m^2}{2} + \tfrac{3}{2}\lambda v^2)h^2 \end{aligned} \tag{6.30}$$

We have that $v = \sqrt{\frac{-m^2}{\lambda}}$ so we know $m^2 + \lambda v^2 = 0$.

$$\begin{aligned} V(\phi) &= [\tfrac{m^2}{2} + \tfrac{3}{2}\lambda(\tfrac{-m^2}{\lambda})]h^2 \\ &= [\tfrac{m^2}{2} - \tfrac{3}{2}m^2]h^2 \end{aligned} \tag{6.31}$$

Finally, we get the result that

$$V(\phi) = -m^2h^2 \tag{6.32}$$

Now that we've considered the potential term, let's consider the kinetic term. We can plug (6.27) in to the first term of (6.14) and solve.

$$\begin{aligned} \tfrac{1}{2}(\partial_\mu\phi)^\dagger(\partial^\mu\phi) &= \tfrac{1}{2}(\partial_\mu(v+h)e^{\frac{i\epsilon}{v}})^\dagger(\partial^\mu(v+h)e^{\frac{i\epsilon}{v}}) \\ &= \tfrac{1}{2}(h'(x)e^{\frac{i\epsilon(x)}{v}} + (v+h(x))\tfrac{i}{v}\epsilon'(x)e^{\frac{i\epsilon(x)}{v}})^\dagger(h'(x)e^{\frac{i\epsilon(x)}{v}} + (v+h(x))\tfrac{i}{v}\epsilon'(x)e^{\frac{i\epsilon(x)}{v}}) \\ &= \tfrac{1}{2}(h'(x)e^{\frac{-i\epsilon(x)}{v}} - (v+h(x))\tfrac{i}{v}\epsilon'(x)e^{\frac{-i\epsilon(x)}{v}})(h'(x)e^{\frac{i\epsilon(x)}{v}} + (v+h(x))\tfrac{i}{v}\epsilon'(x)e^{\frac{i\epsilon(x)}{v}}) \\ &= \tfrac{1}{2}((h'(x))^2 + (\tfrac{(v+h(x))\epsilon'(x)}{v})^2) \\ &= \tfrac{1}{2}((h'(x))^2 + (\epsilon'(x))^2(\tfrac{v^2+2vh(x)+h(x)^2}{v^2})) \\ &= \tfrac{1}{2}((h'(x))^2 + (\epsilon'(x))^2(1 + \tfrac{2h(x)}{v} + (\tfrac{h(x)}{v^2})^2)) \end{aligned} \tag{6.33}$$

Now we can ignore the last two terms here since $\epsilon'(x)^2h(x)$ and $\epsilon'(x)^2h(x)^2$ will both be higher than second order.

$$\begin{aligned} \tfrac{1}{2}(\partial_\mu\phi)^\dagger(\partial^\mu\phi) &= \tfrac{1}{2}((h(x))^2 + (\epsilon'(x))^2) \\ &= \tfrac{1}{2}(\partial_\mu h)^\dagger(\partial^\mu h) + \tfrac{1}{2}(\partial_\mu\epsilon)^\dagger(\partial^\mu\epsilon) \end{aligned} Va \tag{6.34}$$

Substituting (6.32) and (6.34) into our Lagrangian, we find that

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}(\partial_\mu h)^\dagger(\partial^\mu h) + \tfrac{1}{2}(\partial_\mu\epsilon)^\dagger(\partial^\mu\epsilon) - (-m^2h^2) \\ &= \tfrac{1}{2}(\partial_\mu h)^\dagger(\partial^\mu h) - \tfrac{1}{2}(-2m^2)h^2 + \tfrac{1}{2}(\partial_\mu\epsilon)^\dagger(\partial^\mu\epsilon) + O(h^3,\epsilon^3) \end{aligned} \tag{6.35}$$

From this see solution we can see two interesting features. To the order at which we approximated the solution, we are left with two modes. The first is known as the Higgs mode and comes from the terms involving h:

$$\tfrac{1}{2}[(\partial_\mu h)^\dagger(\partial^\mu h) - (-2m^2)h^2] \tag{6.36}$$

The massive Higgs mode has a squared mass equal to $-2m^2$). Oscillations in this field take the Higgs mode up and down the walls of the potential well. The second mode comes from the terms involving $\epsilon$. This is a massless mode, and is called the Nambu-Goldstone mode. Oscillations in this field keep the Nambu-Goldstone mode in the ring of possible minima.

Now that we have a better idea of what symmetry breaking is and what it looks like, let's go back to the context of general relativity.

# 7 Vacuum Solutions in General Relativity

Let's quickly recap what we've done so far. We first talked about field theories, and showed that theories could be written in terms of a Lagrangian. This Lagrangian could be varied, and equations of motion could be found. We then talked about manifolds, and how symmetries can exist. We found that symmetries of the metric could be found using the Killing equation which described the Killing vectors for a space. Finally we talked about spontaneous symmetry breaking. In this discussion we talked about vacuum, or ground state, solutions. In this section we will continue our discussion from section (GR free space). We will choose a certain vacuum solution, and show that it contains some number of symmetries.

First let's look at a few changes to the theory that we introduced when discussing GR in free space. From varying the Lagrangian (4.44) we were able to reproduce the Einstein equations:

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 0 \tag{7.1}$$

This describes a free theory of GR (empty space). We can, however, add energy into the system which will more accurately describe the universe. In this case, Einstein's equations are no longer equal to zero, but instead lead to a term involving an energy density. We would find that:

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{7.2}$$

Where $T_{\mu\nu}$ is the stress-energy tensor, and is an energy density. The $8\pi G$ is a term added to the system which allows for the correct Newtonian limit. We can also consider the case of a dynamic universe. In this case we add a cosmological constant $\Lambda$, and find that the general form of Einstein's equations is:

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi G T_{\mu\nu} \tag{7.3}$$

When we consider this full form of the Einstein equations, we find that the Lagrangian becomes

$$\mathcal{L} = \tfrac{1}{16\pi G}(R - 2\Lambda) \tag{7.4}$$

This Lagrangian has three vacuum solutions, which holds for the ground state solution of the stress-energy tensor, $< T_{\mu\nu} >= 0$. The first is the case where $\Lambda = 0$, which describes a static universe. This is the Minkowski space which we considered before, and describes a "flat" universe. If we allow ourselves to think of this space as two dimensional (where in reality we a considering it to be four-dimensional) flat Minkowski space looks something like Figure 9.

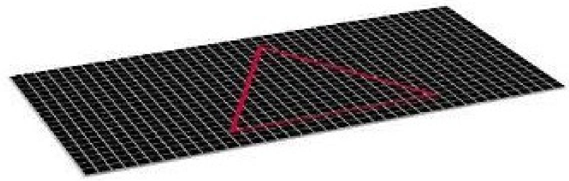

Figure 9: Two-Dimensional Minkowski Space. Figure recreated from Ref. [6].

Another vacuum solution comes from the case where $\Lambda$ is a positive constant. This solution is known as de Sitter space, and describes a dynamic universe which is expanding. Using our two-dimensional analogy, we can think of de Sitter space looking something like Figure 10.

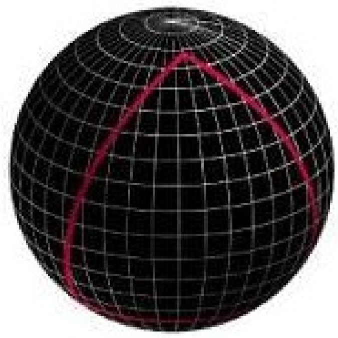

Figure 10: Two-Dimensional de Sitter Space. Figure recreated from Ref. [6].

Finally, the last vacuum solutions comes from the case where $\Lambda$ is a negative constant. Not surprisingly, this describes a dynamic universe which is contracting, and is known as anti-de Sitter space. We can think of a space of this type looking like Figure 11.

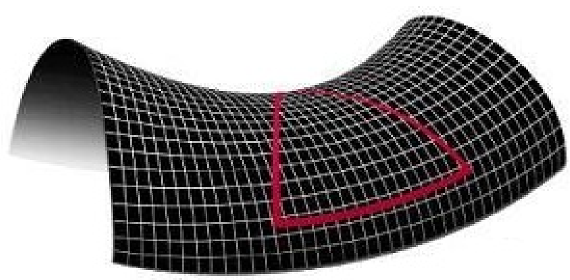

Figure 11: Two-Dimensional Anti-de Sitter Space. Figure recreated from Ref. [6].

In this thesis, we will consider the first of these two vacuum solutions. It should be noted, however, that anti-de Sitter space is equally as rich in content, and should not be disregarded as simply the opposite of a de Sitter space.

### 7.1 Minkowski Space

We'll start our search for symmetries by looking at flat Minkowski space, where our cosmological constant is $\Lambda = 0$. We find that the metric tensor for flat Minkowski space is:

$$< g_{\mu\nu} >= \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.5}$$

If we look at the metric under a diffeomorphism, then we see that

$$g_{\mu\nu} \to g_{\mu\nu} - (\partial_\mu \xi^\alpha) g_{\alpha\nu} - (\partial_\nu \xi^\alpha) g_{\mu\alpha} - \xi^\alpha \partial_\alpha g_{\mu\nu} \tag{7.6}$$

where $\xi^\mu$ is a potential Killing vector. Since our metric is a constant, we can see immediately that we will always lose the last term. Thus, for the vector to be classified as a Killing vector, it must then be true that $g_{\mu\nu} \to g_{\mu\nu}$, or that:

$$(\partial_\mu \xi^\alpha) g_{\alpha\nu} + (\partial_\nu \xi^\alpha) g_{\mu\alpha} = 0 \tag{7.7}$$

We can see that there are a total of 10 independent vectors, $\xi^\mu$, which satisfy the above equation (the Killing equation.) We will first state what they are, and then show that they satisfy the equation. The Killing vectors for Minkowski space are:

$$\xi^\mu = \begin{cases} (1,0,0,0)\varepsilon \\ (0,1,0,0)\varepsilon \\ (0,0,1,0)\varepsilon \\ (0,0,0,1)\varepsilon \\ \epsilon^\mu_\nu x^\nu \end{cases} \tag{7.8}$$

The first four of these Killing vectors are translational; $\varepsilon$ is some small scalar such that $|\varepsilon| \ll 0$. The last equation represents Lorentz transformations, and thus the tensor $\epsilon^\mu_\nu$ is antisymmetric such that $\epsilon^\mu_\nu = -\epsilon^\nu_\mu$, or written out in it's full form:

$$\epsilon^\mu_\nu = \begin{pmatrix} 0 & -\epsilon^1_0 & -\epsilon^2_0 & -\epsilon^3_0 \\ \epsilon^1_0 & 0 & \epsilon^2_1 & \epsilon^3_1 \\ \epsilon^2_0 & -\epsilon^2_1 & 0 & \epsilon^3_2 \\ \epsilon^3_0 & -\epsilon^3_1 & -\epsilon^3_2 & 0 \end{pmatrix} \tag{7.9}$$

The translational Killing vectors are just constants and clearly satisfy equations (7.7). Then, since $\epsilon^\mu_\nu$ is an anti-symmetric tensor, it should be clear that it too satisfies equation (7.7). We have a total of 10 symmetries (isometries) in Minkowski space. There are 4 translational Killing vectors, and 6 Lorentz transformations. From our prior knowledge of special relativity, we know that the Lorentz transformations can be further classified into 3 rotations and 3 Lorentz boosts (the rotations are the Killing vectors with $\epsilon^k_j$ with $j = k = 1, 2, 3$, and the boosts are the Killing vectors with $\epsilon^j_0$.)

### 7.2 de Sitter Space

Now that we understand what the Killing vectors are for Minkowski space, let's go ahead and find the Killing vectors for a de Sitter spacetime.

First, we'll claim that the vacuum metric for a de Sitter spacetime is given by

$$< g_{\mu\nu} >= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{pmatrix} \tag{7.10}$$

where $a(t) = e^{\sqrt{\frac{\Lambda}{3}}t}$. It is verified in Appendix A that this metric does indeed satisfy the Einstein equation. From the rest of this section it will be assumed that this is the metric $g_{\mu\nu}$. In order to more easily write out the full set of Killing vectors for de Sitter space, let's rewrite the Killing equation in a way that is easier to work with.

The equation for Killing vectors with upper indices, $\xi^\mu$, can be written:

$$g_{\alpha\nu}(\partial_\mu \xi^\alpha) + g_{\alpha\mu}(\partial_\nu \xi^\alpha) + \xi^\alpha \partial_\alpha g_{\mu\nu} = 0 \tag{7.11}$$

Let's keep in mind that we can switch between upper indices and lower indices by multiplying by the metric tensor, such that $x^\mu = g^{\mu\nu}\xi_\nu$. We know that the metric must obey the rule that $g_{\mu\nu}g^{\nu\sigma} = \delta^\sigma_\mu$, and so we get that

$$g^{\mu\nu} = (g_{\mu\nu})^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{a(t)^2} & 0 & 0 \\ 0 & 0 & \frac{1}{a(t)^2} & 0 \\ 0 & 0 & 0 & \frac{1}{a(t)^2} \end{pmatrix} \tag{7.12}$$

This gives that

$$\begin{aligned} \xi^0 &= -\xi_0 \\ \xi^j &= \tfrac{1}{a(t)^2}\xi_j \end{aligned} \tag{7.13}$$

Now, we can look at a few representative cases of the Killing equation (7.11), which are when

$$\begin{aligned} &\mu = 0, \nu = 0 \\ &\mu = 0, \nu = j \\ &\mu = j, \nu = k \end{aligned} \tag{7.14}$$

where j and k each run 1, 2, 3. With $\mu = 0, \nu = 0$, the killing equation becomes

$$
\begin{aligned}
g_{\alpha 0}(\partial_0 \xi^\alpha) + g_{\alpha 0}(\partial_0 \xi^\alpha) + \xi^\alpha \partial_\alpha g_{00} &= 0 \\
-(\partial_0 \xi^0) - (\partial_0 \xi^0) &= 0 \\
(\partial_0 \xi_0) + (\partial_0 \xi_0) &= 0 \\
\partial_0 \xi_0 &= 0
\end{aligned} \tag{7.15}
$$

With $\mu = 0, \nu = j$:

$$
\begin{aligned}
g_{\alpha j}(\partial_0 \xi^\alpha) + g_{\alpha 0}(\partial_j \xi^\alpha) + \xi^\alpha \partial_\alpha g_{0j} &= 0 \\
a(t)^2(\partial_0 \xi^j) - (\partial_j \xi^0) &= 0 \\
a(t)^2(\partial_0(\tfrac{1}{a(t)^2}\xi_j)) + (\partial_j \xi_0) &= 0 \\
a(t)^2(-\tfrac{2}{a(t)^3}\dot{a}(t)\xi_j + \tfrac{1}{a(t)^2}\partial_0 \xi_j) + (\partial_j \xi_0) &= 0 \\
\partial_j \xi_0 + \partial_0 \xi_j - 2\tfrac{\dot{a}(t)}{a(t)}\xi_j &= 0
\end{aligned} \tag{7.16}
$$

With $\mu = j, \nu = k$:

$$
\begin{aligned}
g_{\alpha k}(\partial_j \xi^\alpha) + g_{\alpha j}(\partial_k \xi^\alpha) + \xi^\alpha \partial_\alpha g_{jk} &= 0 \\
a(t)^2(\partial_j \xi^k) + a(t)^2(\partial_k \xi^j) + (\xi^0 \partial_0 a(t)^2)\delta_{jk} &= 0 \\
a(t)^2(\partial_j \tfrac{1}{a(t)^2}\xi_k) + a^2(\partial_k \tfrac{1}{a(t)^2}\xi_j) + (\xi^0 \partial_0 a(t)^2)\delta_{jk} &= 0 \\
\partial_j \xi_k + \partial_k \xi_j - (2a(t)\dot{a}(t)\xi_0)\delta_{jk} &= 0
\end{aligned} \tag{7.17}
$$

We now have three equations, which are the final lines of equations (7.15), (7.16), and (7.17). (5.16) is really three separate equations, and (7.17) is really 6 different equations. These equations are expressed explicitly in Appendix B. With these equations in mind, we will state that the Killing vectors for de Sitter space are as follows[1]

$$
\xi^\mu = \begin{cases}
(1, -\sqrt{\frac{\Lambda}{3}}x^1, -\sqrt{\frac{\Lambda}{3}}x^2, -\sqrt{\frac{\Lambda}{3}}x^3) \\
(0,1,0,0) \\
(0,0,1,0) \\
(0,0,0,1) \\
(0, x^2, -x^1, 0) \\
(0, x^3, 0, -x^1) \\
(0, 0, x^3, -x^2) \\
(x^1, \frac{1}{2}\sqrt{\frac{\Lambda}{3}}[(x^1)^2 + (x^2)^2 + (x^3)^2 - \frac{3}{\Lambda}e^{2\sqrt{\frac{\Lambda}{3}}t}] - \sqrt{\frac{\Lambda}{3}}(x^1)^2, -\sqrt{\frac{\Lambda}{3}}x^1x^2, -\sqrt{\frac{\Lambda}{3}}x^1x^3) \\
(x^2, -\sqrt{\frac{\Lambda}{3}}x^2x^1, \frac{1}{2}\sqrt{\frac{\Lambda}{3}}[(x^1)^2 + (x^2)^2 + (x^3)^2 - \frac{3}{\Lambda}e^{2\sqrt{\frac{\Lambda}{3}}t}] - \sqrt{\frac{\Lambda}{3}}(x^2)^2, -\sqrt{\frac{\Lambda}{3}}x^2x^3) \\
(x^3, -\sqrt{\frac{\Lambda}{3}}x^3x^1, -\sqrt{\frac{\Lambda}{3}}x^3x^2, \frac{1}{2}\sqrt{\frac{\Lambda}{3}}[(x^1)^2 + (x^2)^2 + (x^3)^2 - \frac{3}{\Lambda}e^{2\sqrt{\frac{\Lambda}{3}}t}] - \sqrt{\frac{\Lambda}{3}}(x^3)^2)
\end{cases} \tag{7.18}
$$

[1] These Killing vectors were deduced from Ref. [2], page 136.

We can note that although these have a more complicated form than the Killing vectors of Minkowski space, they are actually quite similar. The first four are de Sitter translations; the three spatial translations are the same as those for Minkowski space, and the time-translation is now of a slightly different form. The other six Killing vectors are de Sitter Lorentz transformations; vectors 5-7 are essentially de Sitter rotations and vectors 8-10 are essentially de Sitter Lorentz boosts. In Appendix B we go through the details of showing that the 10 Killing vectors of (7.18) are in fact solutions to the Killing equation.

For a more complete discussion of de Sitter spaces, see Ref. [4] and Ref. [2].

# 8 The Bumblebee Model

Now that we have determined the Killing vectors for both Minkowski space and de Sitter space, we will introduce a model in which we can allow spontaneous symmetry breaking to occur. We will look at how the total number of symmetries for a given theory changes when a Lorentz symmetry is spontaneously broken. This model is called the Bumblebee Model, and is a combination of Einstein's GR, Maxwell theory, and a vector potential field. The Lagrangian for the Bumblebee Model is

$$\mathcal{L} = \frac{1}{16\pi G}(R - 2\Lambda) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(B_\mu B^\mu + b^2) \tag{8.1}$$

with

$$V(B_\mu B^\mu + b^2) = \tfrac{1}{2}\kappa(B_\mu B^\mu + b^2)^2 \tag{8.2}$$

where $\kappa$ is some constant. The Bumblebee Model gets it's name from the $B_\mu$ terms in the vector potential. We can see that the first term is Einstein's GR, and that the second term is Maxwell's theory. If these were the only two terms in the Lagrangian then we would refer to the theory as Einstein-Maxwell theory. The last term is similar to some of the potential terms which we have seen earlier. There is a symmetry here in that the vacuum solution of the potential can have an infinite number of solutions; as long as it is true that $B_\mu B^\mu = -b^2$, the potential will be zero. However, from our early study of the spontaneous symmetry breaking mechanism, we know that the act of actually choosing one of these solutions breaks symmetry.

## 8.1 Minkowski Space Revisited

Let's first consider the case of a flat Minkowski space, where the cosmological constant is $\Lambda = 0$. We know that the Killing vectors for this space are those given in equation (5.8). Let's go ahead and (spontaneously) choose a vacuum solution to be $< B_\mu >= b_\mu$ such that $b_\mu b^\mu = -b^2$. For example, this could be

$$< B_\mu >= b_\mu = (b, 0, 0, 0) \tag{8.3}$$

By choosing this solution, we have spontaneously broken Lorentz symmetry. We can verify that this is actually a vacuum solution for the system. We know that the strength tensor is

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu = 0 \tag{8.4}$$

because $B_\mu$ is a constant. So, the second term is at a minimum. We can verify that this is also a vacuum solution for the first term by looking at the Einstein equation, $G_{\mu\nu} = 4\pi G T_{\mu\nu}$. We know that

$$T_{\mu\nu} = F_{\mu\nu} F^\alpha_\nu - \tfrac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - V g_{\mu\nu} + 2V' B_\mu B_\nu \tag{8.5}$$

We showed above that the first two terms go to zero. We also know that the vector potential $V(B^\mu B_\mu + b^2)$ is of the form $V = \frac{1}{2}\kappa(B^\mu B_\mu + b^2)^2$ so V' is of the form $\kappa(B^\mu B_\mu + b^2)$. With our choice of vacuum vector $B^\mu B_\mu + b^2$ is zero, so the second two terms also go to zero. Thus, $G_{\mu\nu} = 0$, and $b_\mu$ is in fact a vacuum solution.

Now let's look at what happens to the vacuum vector $b_\mu$ under an isometry. The vacuum vector will take the form:

$$b_\mu \to b_\mu - (\partial_\mu \xi^\alpha) b_\alpha - \xi^\alpha(\partial_\alpha b_\mu) \tag{8.6}$$

Now, our vacuum vector is constant, and for our translational isometries $\xi^\mu$ is constant, so we can readily see that $b_\mu \to b_\mu$ and thus the symmetry is unbroken. For our 6 Lorentz transformations, we see that

$$\begin{aligned} b_\mu &\to b_\mu - (\partial_\mu \epsilon^\alpha_\nu x^\nu) b_\alpha \\ &= b_\mu - (\partial_\mu \epsilon^0_\nu x^\nu) b_0 \\ &= b_\mu - (\partial_\mu \epsilon^0_j x^j) b \\ &= b_\mu - \epsilon^0_j \delta^j_\mu b \\ &= b_\mu - \epsilon^0_\mu b \end{aligned} \tag{8.7}$$

We know that the Lorentz transformations have the form

$$\begin{aligned} \Lambda^\nu_\mu &= \delta^\nu \mu + \epsilon^\nu_\mu \\ &= \begin{pmatrix} 1 & -\epsilon^1_0 & -\epsilon^2_0 & -\epsilon^3_0 \\ \epsilon^1_0 & 1 & \epsilon^2_1 & \epsilon^3_1 \\ \epsilon^2_0 & -\epsilon^2_1 & 1 & \epsilon^3_2 \\ \epsilon^3_0 & -\epsilon^3_1 & -\epsilon^3_2 & 1 \end{pmatrix} \end{aligned} \tag{8.8}$$

So, we have 3 rotations (which are $\epsilon^k_j$, with j=k=1,2,3) and 3 boosts (which are $\epsilon^j_0$.) Since the diffeomorphism for the Lorentz transformation has the form $b_\mu \to b_\mu - \epsilon^0_\mu b$, the 3 rotations clearly give $b_\mu \to b_\mu$, and thus leave the symmetry unbroken. For a boost in the first spatial direction, the transformation matrix would become

$$\Lambda^\nu_\mu = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8.9}$$

For a boost in the second spatial direction, the $\epsilon_0^2$ term would get the $-\gamma v$ term, and so on. For the three boosts, we see that

$$\begin{aligned} b_0 &\to b_0 \\ b_1 &\to b_1 - \epsilon_1^0 b \\ b_2 &\to b_2 - \epsilon_2^0 b \\ b_3 &\to b_3 - \epsilon_3^0 b \end{aligned} \tag{8.10}$$

We started with a total of ten isometries for Minkowski space, four translations and six Lorentz transformations. We see that under differomorphisms, these ten symmetries reduce to seven symmetries, four translations and three Lorentz transformations (the rotations.) The other three Lorentz transformations (the boosts) spontaneously break the symmetry of the system.

## 8.2 de Sitter Space Revisited

Now let's look at the case where we have a positive cosmological constant ($\Lambda > 0$). A vacuum solution for the metric in de Sitter space is (as we showed before)

$$< g_{\mu\nu} >= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{pmatrix} \tag{8.11}$$

with $a(t) = e^{\sqrt{\frac{\Lambda}{3}}t}$. We know that there are still ten isometries, each represented by a Killing vector $\xi^\mu$ (which are given explicitly in Appendix B.) Let's again consider the case where $< B_\mu >= b_\mu = (b, 0, 0, 0)$. Just as in the Minkowski case, we know that this choice of vacuum solution spontaneously breaks symmetry in the potential term, and is a vacuum solution for the whole system. We want to again look at the vacuum vector under a diffeomorphism using the ten Killing vectors for de Sitter space. We know that in general

$$b_\mu \to b_\mu - (\partial_\mu \xi^\alpha) b_\alpha - \xi^\alpha (\partial_\alpha b_\mu) \tag{8.12}$$

Again, our vacuum vector is a constant, so the last term will always be zero. We are then able to say that the symmetry is unbroken if $(\partial_\mu \xi^\alpha) a_\alpha$ is zero. We find that

$$\begin{aligned} (\partial_\mu \xi^\alpha) b_\alpha &= (\partial_\mu \xi^0) b_0 + (\partial_\mu \xi^1) b_1 + (\partial_\mu \xi^2) b_2 + (\partial_\mu \xi^3) b_3 \\ &= (\partial_\mu \xi^0) b_0 \end{aligned} \tag{8.13}$$

If the first component of the Killing vector is a constant, we can say immediately that the vacuum vector will be invariant under the diffeomorphism, and thus the symmetry will be unbroken. This applies to seven of our killing vectors, written below:

$$\xi^\mu = \begin{cases} (1, -\sqrt{\frac{\Lambda}{3}}x^1, -\sqrt{\frac{\Lambda}{3}}x^2, -\sqrt{\frac{\Lambda}{3}}x^3) \\ (0,1,0,0) \\ (0,0,1,0) \\ (0,0,0,1) \\ (0,x^2,-x^1,0) \\ (0,x^3,0,-x^1) \\ (0,0,x^3,-x^2) \end{cases} \tag{8.14}$$

The other three killing vectors have spatial dependence in their first components. They are

$$\xi^\mu = \begin{cases} (x^1, \frac{1}{2}\sqrt{\frac{\Lambda}{3}}[(x^1)^2 + (x^2)^2 + (x^3)^2 - \frac{3}{\Lambda}e^{2\sqrt{\frac{\Lambda}{3}}t}] - \sqrt{\frac{\Lambda}{3}}(x^1)^2, -\sqrt{\frac{\Lambda}{3}}x^1x^2, -\sqrt{\frac{\Lambda}{3}}x^1x^3) \\ (x^2, -\sqrt{\frac{\Lambda}{3}}x^2x^1, \frac{1}{2}\sqrt{\frac{\Lambda}{3}}[(x^1)^2 + (x^2)^2 + (x^3)^2 - \frac{3}{\Lambda}e^{2\sqrt{\frac{\Lambda}{3}}t}] - \sqrt{\frac{\Lambda}{3}}(x^2)^2, -\sqrt{\frac{\Lambda}{3}}x^2x^3) \\ (x^3, -\sqrt{\frac{\Lambda}{3}}x^3x^1, -\sqrt{\frac{\Lambda}{3}}x^3x^2, \frac{1}{2}\sqrt{\frac{\Lambda}{3}}[(x^1)^2 + (x^2)^2 + (x^3)^2 - \frac{3}{\Lambda}e^{2\sqrt{\frac{\Lambda}{3}}t}] - \sqrt{\frac{\Lambda}{3}}(x^3)^2) \end{cases} \tag{8.15}$$

We're only worried about their first components (according to equation (6.13)), so let's write them as

$$\xi^\mu = (x^k, \dots) \tag{8.16}$$

where j=1,2,3. We can see then that

$$(\partial_\mu \xi^0) b_0 = \delta^k_\mu b \tag{8.17}$$

We find that $b_\mu \to b_\mu - \delta^k_\mu b$ for the killing vector $\xi^\mu = (x^k, \dots)$.

We started with a total of ten isometries and ten Killing vectors for de Sitter space, broken down into four de Sitter translations and six de Sitter Lorentz transformations. After looking at how a vacuum vector transforms under a diffeomorphism, we find that we are left with only seven isometries, which are the four de Sitter translations, and the three de Sitter rotations. The other three de Sitter Lorentz transformations (the de Sitter boosts) spontaneously break the symmetry of the system. This is similar to the symmetries broken in Minkowski space.

Now let's consider a different vacuum solution, which now is time-dependent. Let's say that our timelike vacuum vector is now spontaneously chosen to be

$$< B_\mu >= b_\mu(t) = b(\frac{\sqrt{a(t)^2+1}}{a(t)}, e_j) \tag{8.18}$$

where $e_j$ is a unit spatial vector obeying the rule $e_j \cdot e_j = 1$. Since we are still working in a de Sitter spacetime, we know that we can use the same metric as before. We can then verify that the above proposed vacuum vector is in fact a vacuum solution. The potential term should be at a minimum, which will occur when $B_\mu B^\mu = -b^2$.

$$
\begin{aligned}
< B_\mu >< B^\mu > &= b_\mu b^\mu = b_\mu g^{\mu\nu} b_\nu \\
&= -b^2 \tfrac{a(t)^2+1}{a(t)^2} + \tfrac{1}{a(t)^2} b^2 (e_j \cdot e_j) \\
&= -b^2 [1 + \tfrac{1}{a(t)^2} - \tfrac{1}{a(t)^2}] \\
&= -b^2
\end{aligned} \tag{8.19}
$$

So the potential is at a minimum. We can also show that the strength tensor $F_{\mu\nu}$ is zero.

$$F_{0j} = \partial_0 b_j - \partial_j b_0 = 0 \tag{8.20}$$

since $b_j$ has no time-dependence and $b_0$ has no spatial-dependence, and

$$F_{jk} = \partial_j b_k - \partial_k b_j = 0 \tag{8.21}$$

since $b_j$ has no spatial-dependence either. Now that we know that the strength tensor is zero and that the potential is zero, we know that this vacuum vector is a vacuum solution for the GR term in the Bumblebee Lagrangian. Thus, our time-dependent vacuum vector is in fact a vacuum solution for the system. Let's now check how the vacuum vector transforms under our set of ten isometries. We know that it is true that

$$b_\mu \to b_\mu - (\partial_\mu \xi^\alpha) b_\alpha - \xi^\alpha (\partial_\alpha b_\mu) \tag{8.22}$$

Let's first consider $b_j$. These terms are all constants, so we lose our $\partial_\alpha b_\mu$ term. Now let's look at $b_0$. We first see that there is no time-dependence in the first component of any of our killing vectors. It can also be seen that $b_0$ has only time-dependence with no spatial-dependence. Thus we get that

$$
\begin{aligned}
&b_j \to b_j - (\partial_j \xi^\mu) b_\mu \\
&b_0 \to b_0 - (\partial_0 \xi^j) b_j - \xi^0 (\partial_0 b_0)
\end{aligned} \tag{8.23}
$$

We can now look at our first isometry, represented by the Killing vector:

$$\xi^\mu_{(1)} = (1, -\sqrt{\tfrac{\Lambda}{3}} x^1, -\sqrt{\tfrac{\Lambda}{3}} x^2, -\sqrt{\tfrac{\Lambda}{3}} x^3) \tag{8.24}$$

We see immediately that:

$$b_j \to b_j + \sqrt{\tfrac{\Lambda}{3}} \delta^k_j e_k \tag{8.25}$$

and, since $\xi^j$ has no time dependance:

$$b_0 \to b_0 - \partial_0 b_0 \tag{8.26}$$

We can then compute that time derivative of $b_0$ (which will continue to appear)

$$
\begin{aligned}
\partial_0 b_0 &= \tfrac{d}{dt}[b\tfrac{\sqrt{a(t)^2+1}}{a(t)}] \\
&= b[\tfrac{1}{a(t)}\tfrac{d}{dt}\sqrt{a(t)^2+1} + \sqrt{a(t)^2+1}\tfrac{d}{dt}\tfrac{1}{a(t)}] \\
&= b[\tfrac{1}{a(t)}\tfrac{1}{2}\tfrac{1}{\sqrt{a(t)^2+1}}(2a(t)\dot{a}(t)) - \sqrt{a(t)^2+1}\tfrac{\dot{a}(t)}{a(t)^2}] \\
&= b[\tfrac{\dot{a}(t)}{\sqrt{a(t)^2+1}} - \tfrac{\sqrt{a(t)^2+1}\dot{a}(t)}{a(t)^2}] \\
&= b[\tfrac{\dot{a}(t)a(t)^2}{\sqrt{a(t)^2+1}a(t)^2} - \tfrac{(a(t)^2+1)\dot{a}(t)}{\sqrt{a(t)^2+1}a(t)^2}] \\
&= b[\tfrac{\dot{a}(t)a(t)^2 - a(t)^2\dot{a}(t) - \dot{a}(t)}{a(t)^2\sqrt{a(t)^2+1}}] \\
&= -\tfrac{b\dot{a}(t)}{a(t)^2\sqrt{a(t)^2+1}}
\end{aligned}
\tag{8.27}
$$

So we see that:

$$
b_0 \to b_0 + \tfrac{b\dot{a}(t)}{a(t)^2\sqrt{a(t)^2+1}} \tag{8.28}
$$

For the first isometry, the symmetry is broken. Let's now look at the next three isometries, given by the Killing vectors

$$
\begin{aligned}
\xi^\mu_{(2)} &= (0,1,0,0) \\
\xi^\mu_{(3)} &= (0,0,1,0) \\
\xi^\mu_{(4)} &= (0,0,0,1)
\end{aligned}
\tag{8.29}
$$

Looking at $b_j$, we see that there is no spatial-dependence in $\xi^\mu$, so we lose the second term. In looking at $b_0$, we see that $\xi^j$ has no time-dependence and that $\xi^0$ is zero, so we lose the second and third terms. Thus we can say that $b_\mu \to b_\mu$ for the Killing vectors $\xi^\mu_{(2)}$, $\xi^\mu_{(3)}$, and $\xi^\mu_{(4)}$, and the symmetry is unbroken.

Let's now look at the next three Killing vectors. For the vector $\xi^\mu = (0, x^2, -x^1, 0)$, we get that

$$
\begin{aligned}
b_1 &\to b_1 + b_2 \\
b_2 &\to b_2 - b_1 \\
b_3 &\to b_3 \\
b_0 &\to b_0
\end{aligned}
\tag{8.30}
$$

For the vector $\xi^\mu = (0, x^3, 0, -x^1)$, we get that

$$
\begin{aligned}
b_1 &\to b_1 + b_3 \\
b_2 &\to b_2 \\
b_3 &\to b_3 - b_1 \\
b_0 &\to b_0
\end{aligned}
\tag{8.31}
$$

For the vector $\xi^\mu = (0, 0, x^3, -x^2)$, we get that

$$\begin{aligned} b_1 &\to b_1 \\ b_2 &\to b_2 + b_3 \\ b_3 &\to b_3 - b_2 \\ b_0 &\to b_0 \end{aligned} \tag{8.32}$$

While at first it may appear that these vacuum vectors are not invariant under the isometries, this is not the case. It will always be true that it is possible to find a frame in which the above vacuum vectors are invariant. For example, let's look at the transformations of equation (8.30). We can choose to look at the frame where our vacuum vector is $b_\mu(t) = b(\frac{\sqrt{a(t)^2+1}}{a(t)}, 0, 0, 1)$. In this frame, we still obey the rule that $e_j \cdot e_j = 1$ since here $e_j = (0, 0, 1)$ and $(0, 0, 0) \cdot (0, 0, 1) = 1$. We also see that $b_1 = b_2 = 0$, and so from equation (6.30)

$$\begin{aligned} b_1 = 0 &\to 0 + 0 = 0 \\ b_2 = 0 &\to 0 - 0 = 0 \\ b_3 = b &\to b \\ b_0 = b\frac{\sqrt{a(t)^2+1}}{a(t)} &\to b\frac{\sqrt{a(t)^2+1}}{a(t)} \end{aligned} \tag{8.33}$$

So there exists a frame where the vacuum vector is invariant under the isometry. This is also true for the other two de Sitter-rotational Killing vectors. Thus, the symmetry is unbroken for all three of these isometries.

Let's now consider the next isometry, which we can then use to include the ninth and tenth isometries as well. This symmetry is given by the Killing vector:

$$\xi^\mu_{(5)} = (x^1, \tfrac{1}{2}\sqrt{\tfrac{\Lambda}{3}}[(x^1)^2 + (x^2)^2 + (x^3)^2 - \tfrac{3}{\Lambda}e^{2\sqrt{\frac{\Lambda}{3}}t}] - \sqrt{\tfrac{\Lambda}{3}}(x^1)^2, -\sqrt{\tfrac{\Lambda}{3}}x^1x^2, -\sqrt{\tfrac{\Lambda}{3}}x^1x^3) \tag{8.34}$$

Let's consider each component of the $b_\mu$ vector separately.

$$\begin{aligned} b_1 &\to b_1 - (\partial_1\xi^\mu)b_\mu \\ b_1 &\to b_1 - [(\partial_1\xi^0)b_0 + (\partial_1\xi^1)b_1 + (\partial_1\xi^2)b_2 + (\partial_1\xi^3)b_3] \\ b_1 &\to b_1 - [b_0 + (\sqrt{\tfrac{\Lambda}{3}}x^1 - 2\sqrt{\tfrac{\Lambda}{3}}x^1)b_1 - \sqrt{\tfrac{\Lambda}{3}}x^2 b_2 - \sqrt{\tfrac{\Lambda}{3}}x^3 b_3] \end{aligned} \tag{8.35}$$

$$\boxed{b_1 \to b_1 - b\frac{\sqrt{a(t)^2+1}}{a(t)} + \sqrt{\tfrac{\Lambda}{3}}x^j e_j} \tag{8.36}$$

$$\begin{aligned} b_2 &\to b_2 - [(\partial_2\xi^0)b_0 + (\partial_2\xi^1)b_1 + (\partial_2\xi^2)b_2 + (\partial_2\xi^3)b_3] \\ b_2 &\to b_2 - [\sqrt{\tfrac{\Lambda}{3}}x^2 b_1 - \sqrt{\tfrac{\Lambda}{3}}x^1 b_2] \end{aligned} \tag{8.37}$$

$$\boxed{b_2 \to b_2 - \sqrt{\tfrac{\Lambda}{3}}(x^2 e_1 - x^1 e_2)} \tag{8.38}$$

$$\begin{aligned} &b_3 \to b_3 - [(\partial_3\xi^0)b_0 + (\partial_3\xi^1)b_1 + (\partial_3\xi^2)b_2 + (\partial_3\xi^3)b_3] \\ &b_3 \to b_3 - [\sqrt{\tfrac{\Lambda}{3}}x^3 b_1 - \sqrt{\tfrac{\Lambda}{3}}x^1 b_3] \end{aligned} \tag{8.39}$$

$$\boxed{b_3 \to b_3 - \sqrt{\tfrac{\Lambda}{3}}(x^3 e_1 - x^1 e_3)} \tag{8.40}$$

$$\begin{aligned} &b_0 \to b_0 - (\partial_0\xi^j)b_j - \xi^0(\partial_0 b_0) \\ &b_0 \to b_0 - [(\partial_0\xi^1)b_1 + (\partial_0\xi^2)b_2 + (\partial_0\xi^3)b_3] - \xi^0(\partial_0 b_0) \\ &b_0 \to b_0 - [-\tfrac{1}{2}\sqrt{\tfrac{\Lambda}{3}}\tfrac{3}{\Lambda}2\sqrt{\tfrac{\Lambda}{3}}e^{2\sqrt{\frac{\Lambda}{3}}t}]b_1 + x^1\tfrac{b\dot{a}}{a^2\sqrt{a^2+1}} \end{aligned} \tag{8.41}$$

$$\boxed{b_0 \to b_0 + e^{2\sqrt{\frac{\Lambda}{3}}t}e_1 + x^1\tfrac{b\dot{a}}{a^2\sqrt{a^2+1}}} \tag{8.42}$$

By inspection we can see that this can generalize to include the ninth and tenth Killing vectors, which are similar in form. If we abbreviate these Killing vectors as $\xi^\mu = (x^k, \dots)$, then we get that:

$$\begin{aligned} &b_j \to b_j - \delta_j^k[b\tfrac{\sqrt{a(t)^2+1}}{a(t)} - \sqrt{\tfrac{\Lambda}{3}}x^j e_j] - \delta_j^{i\neq k}[\tfrac{\Lambda}{3}(x^i e_k - x^k e_i)] \\ &b_0 \to b_0 + e^{2\sqrt{\frac{\Lambda}{3}}t}e_k + x^k\tfrac{b\dot{a}(t)}{a(t)^2\sqrt{a(t)^2+1}} \end{aligned} \tag{8.43}$$

Clearly, these last three de Sitter Lorenz boosts have broken the symmetry of the system. We have seen that in de Sitter space, if we choose a time-dependent vacuum solution, we are left with only 6 of the original 10 isometries which existed for the space.

So, why might this result of being left with only 6 of our original ten symmetries be important? We can find an answer by looking at how our actual universe has been observed to behave. Through many astronomical observations, such as those looking at the Cosmic Microwave Background radiation, it has been found that our universe is very nearly spatially homogeneous and isotropic. Isotropy means that at specific point in the universe, space will look the same in any given direction. More formally, we can say that a manifold $M$ is isotropic around a point $p$ if for two vectors, $A$ and $B$, in the tangent plane at the point $p$ ($T_pM$) there is an isometry on $M$ which gives that the pushforward of the vector $B$ under the isometry is parallel with the vector $A$. Homogeneity means that the metric is the same throughout the entire space (on the entire manifold). Again more formally, this means that if we look at any two points $a$ and $b$ on a manifold $M$, there will be an isometry which takes $a$ into $b$.

These two features are the same features which are assumed to be true in the Friedmann-Robertson-Walker (FRW) solution to the Einstein equation. It turns out that in this solution the universe contains exactly six isometries; there are three spatial translations which come from the homogeneity condition, and three spatial rotations which come form the isotropy condition. This is exactly the number of isometries which we found using our Bumblebee Model with a time-dependent vacuum solution. This suggests that if our universe could be permeated by a vector field, such as that in our Bumblebee model which spontaneously breaks Lorentz symmetry, then a FRW homogeneous and isotropic solution could exist for the vacuum itself (assuming a universe with no matter.)

# 9 Summary and Conclusion

At the beginning of this thesis we sought to find out what happens to the usual symmetries of General Relativity when we look at a specific model (the Bumblebee Model,) where we force spontaneous Lorentz violation to occur. In order to answer this question we had some work to do. First, we needed to learn about how it could be possible to express GR as a field theory. Then, since the mathematics of GR all take place on curved spaces, we had to become familiar working with manifolds, which are the mathematical objects used to talk about curved spaces. We saw how symmetries could exist on a manifold, and also saw how the spontaneous symmetry breaking mechanism occurs. We then shifted our focus back to GR, and looked at different vacuum solutions for the theory. Finally, we were able to introduce the Bumblebee Model, and look at how symmetries of the theory as a whole were broken when we have the vector potential term which spontaneous breaks Lorentz symmetry.

We found that when we look at the theory in Minkowski space (with a cosmological constant equal to zero and thus no curvature, which describes a static universe,) we have a total of ten symmetries, or isometries, each represented by a Killing vector. When we choose a constant vacuum solution which spontaneously breaks Lorentz symmetry in the vector potential term, we lose three of our symmetries (the three Lorentz tansformation boosts.) We then went on to look at de Sitter space where we have a positive cosmological constant, which corresponds to an expanding universe. Here, we found that again we have a total of ten symmetries represented by a set of ten Killing vectors. When we spontaneously break Lorentz symmetry in the vector potential using the same constant vacuum solution that we used for Minkowski space, we found that we again lost three symmetries, which were the three de Sitter Lorentz boosts.

Finally we looked at the case where we spontaneously broke Lorentz symmetry by choosing a time-dependent vacuum solution. In this case, we broke the same three symmetries as with the constant vacuum solution, plus one more (the time-translation Killing vector.) This left us with a total of six Killing vectors, out of the original set of ten. This happens to be the same number of symmetries which our actual universe has been observed to possess, and which are described by the FRW solution to the Einstein equation. Therefore it could well be the case that our universe is permeated by a vector field similar in nature to the one given in the Bumblebee Model.

# A Verifying the Metric for de Sitter Space

The metric for de Sitter space is the Friedmann-Lematre-Robertson-Walker metric, which is given as:

$$< g_{\mu\nu} >= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{pmatrix} \tag{A.1}$$

with $a = e^{\sqrt{\frac{\Lambda}{3}}t}$. The Einstein equation for a universe with an energy density $T_{\mu\nu}^{(vac)}$ is given by

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}^{(vac)} \tag{A.2}$$

where the (vacuum) energy density has the form

$$T_{\mu\nu}^{(vac)} = \begin{pmatrix} \frac{\Lambda}{8\pi G} & 0 & 0 & 0 \\ 0 & -\frac{\Lambda}{8\pi G} a(t)^2 & 0 & 0 \\ 0 & 0 & -\frac{\Lambda}{8\pi G} a(t)^2 & 0 \\ 0 & 0 & 0 & -\frac{\Lambda}{8\pi G} a(t)^2 \end{pmatrix} \tag{A.3}$$

We know that the Ricci tensor is

$$R_{\mu\nu} = R^{\kappa}_{\mu\kappa\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \tfrac{1}{2}(\partial_\mu \Gamma^\alpha{}_{\nu\alpha} + \partial_\nu \Gamma^\alpha{}_{\mu\alpha}) + \Gamma^\beta{}_{\beta\alpha} \Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\mu\beta} \Gamma^\beta{}_{\nu\alpha} \tag{A.4}$$

The only nonzero connection coefficients are

$$\begin{aligned} \Gamma^j_{k0} &= \tfrac{\dot{a}(t)}{a(t)} \delta^j_k \\ \Gamma^0_{jk} &= \dot{a}(t) a(t) \delta_{jk} \end{aligned} \tag{A.5}$$

This yields that the only nonzero components of the Ricci tensor $R_{\mu\nu}$ are

$$\begin{aligned} R_{00} &= -3 \tfrac{\ddot{a}(t)}{a(t)} \\ R_{jj} &= 2\dot{a}(t)^2 + a(t)\ddot{a}(t) \end{aligned} \tag{A.6}$$

Likewise, the Ricci scalar is given by

$$R = 6[\tfrac{\ddot{a}(t)}{a(t)} + (\tfrac{\dot{a}(t)}{a(t)})^2] \tag{A.7}$$

Since we have that $a = e^{\sqrt{\frac{\Lambda}{3}}t}$, we can find all necessary terms involving $a(t)$. We get that

$$
\begin{aligned}
\dot{a}(t) &= \sqrt{\tfrac{\Lambda}{3}}e^{\sqrt{\frac{\Lambda}{3}}t} \\
\ddot{a}(t) &= \tfrac{\Lambda}{3}e^{\sqrt{\frac{\Lambda}{3}}t} \\
\tfrac{\dot{a}(t)}{a(t)} &= \sqrt{\tfrac{\Lambda}{3}} \\
\tfrac{\ddot{a}(t)}{a(t)} &= \tfrac{\Lambda}{3} \\
a(t)\ddot{a}(t) &= \tfrac{\Lambda}{3}e^{2\sqrt{\frac{\Lambda}{3}}t}
\end{aligned}
\tag{A.8}
$$

Solving for $R_{00}$ gives that

$$
R_{00} = -3\tfrac{\Lambda}{3} \tag{A.9}
$$

Solving for $R_{jj}$ gives that

$$
\begin{aligned}
R_{jj} &= 2(\sqrt{\tfrac{\Lambda}{3}}e^{\sqrt{\frac{\Lambda}{3}}t})^2 + \tfrac{\Lambda}{3}e^{2\sqrt{\frac{\Lambda}{3}}t} \\
&= 2\tfrac{\Lambda}{3}e^{2\sqrt{\frac{\Lambda}{3}}t} + \tfrac{\Lambda}{3}e^{2\sqrt{\frac{\Lambda}{3}}t} \\
&= \Lambda e^{2\sqrt{\frac{\Lambda}{3}}t}
\end{aligned}
\tag{A.10}
$$

This gives that the full form of the Ricci tensor is

$$
R_{\mu\nu} = \begin{pmatrix} -\Lambda & 0 & 0 & 0 \\ 0 & \Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 & 0 \\ 0 & 0 & \Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 \\ 0 & 0 & 0 & \Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} \end{pmatrix} \tag{A.11}
$$

The Ricci scalar is found as

$$
\begin{aligned}
R &= 6[\tfrac{\Lambda}{3} + (\sqrt{\tfrac{\Lambda}{3}})^2] \\
&= 6[\tfrac{2\Lambda}{3}] \\
&= 4\Lambda
\end{aligned}
\tag{A.12}
$$

Now that we have the Ricci tensor and Ricci scalar, we see that the left hand side of equation (A.2) is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} =$$

$$\begin{pmatrix} -\Lambda & 0 & 0 & 0 \\ 0 & \Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 & 0 \\ 0 & 0 & \Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 \\ 0 & 0 & 0 & \Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} \end{pmatrix} - \frac{1}{2}(4\Lambda) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 & 0 \\ 0 & 0 & e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 \\ 0 & 0 & 0 & e^{2\sqrt{\frac{\Lambda}{3}}t} \end{pmatrix}$$

$$= \begin{pmatrix} -\Lambda & 0 & 0 & 0 \\ 0 & \Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 & 0 \\ 0 & 0 & \Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 \\ 0 & 0 & 0 & \Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} \end{pmatrix} - \begin{pmatrix} -2\Lambda & 0 & 0 & 0 \\ 0 & 2\Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 & 0 \\ 0 & 0 & 2\Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 \\ 0 & 0 & 0 & 2\Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} \end{pmatrix}$$

$$= \begin{pmatrix} \Lambda & 0 & 0 & 0 \\ 0 & -\Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 & 0 \\ 0 & 0 & -\Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 \\ 0 & 0 & 0 & -\Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} \end{pmatrix} \tag{A.13}$$

The right hand side of equation (A.2) is

$$\begin{aligned} &8\pi G \begin{pmatrix} \frac{\Lambda}{8\pi G} & 0 & 0 & 0 \\ 0 & -\frac{\Lambda}{8\pi G}a(t)^2 & 0 & 0 \\ 0 & 0 & -\frac{\Lambda}{8\pi G}a(t)^2 & 0 \\ 0 & 0 & 0 & -\frac{\Lambda}{8\pi G}a(t)^2 \end{pmatrix} \\ =&8\pi G \begin{pmatrix} \frac{\Lambda}{8\pi G} & 0 & 0 & 0 \\ 0 & -\frac{\Lambda}{8\pi G}e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 & 0 \\ 0 & 0 & -\frac{\Lambda}{8\pi G}e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 \\ 0 & 0 & 0 & -\frac{\Lambda}{8\pi G}e^{2\sqrt{\frac{\Lambda}{3}}t} \end{pmatrix} \\ =& \begin{pmatrix} \Lambda & 0 & 0 & 0 \\ 0 & -\Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 & 0 \\ 0 & 0 & -\Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} & 0 \\ 0 & 0 & 0 & -\Lambda e^{2\sqrt{\frac{\Lambda}{3}}t} \end{pmatrix} \end{aligned} \tag{A.14}$$

Comparing equations (A.13) and (A.14), we see that the left hand side and right hand sides of (A.2) are the same, and thus the Friedmann-Lematre-Robertson-Walker metric is a solution to the Einstein equation.

## B Verifying the Killing Vectors in de Sitter Space

For reference we'll reproduce the Killing equation expanded to 10 equations, and the 10 proposed Killing vectors below. Here we will assume that the quantity $a$ is time-dependent such that $a = a(t)$.

$$\partial_0 \xi_0 = 0 \tag{B.1a}$$
$$\partial_1 \xi_0 + \partial_0 \xi_1 - 2\frac{\dot{a}}{a}\xi_1 = 0 \tag{B.1b}$$
$$\partial_2 \xi_0 + \partial_0 \xi_2 - 2\frac{\dot{a}}{a}\xi_2 = 0 \tag{B.1c}$$
$$\partial_3 \xi_0 + \partial_0 \xi_3 - 2\frac{\dot{a}}{a}\xi_3 = 0 \tag{B.1d}$$
$$\partial_1 \xi_1 - a\dot{a}\xi_0 = 0 \tag{B.1e}$$
$$\partial_2 \xi_2 - a\dot{a}\xi_0 = 0 \tag{B.1f}$$
$$\partial_3 \xi_3 - a\dot{a}\xi_0 = 0 \tag{B.1g}$$
$$\partial_1 \xi_2 + \partial_2 \xi_1 = 0 \tag{B.1h}$$
$$\partial_1 \xi_3 + \partial_3 \xi_1 = 0 \tag{B.1i}$$
$$\partial_2 \xi_3 + \partial_3 \xi_2 = 0 \tag{B.1j}$$

$$\begin{aligned}
\xi^\mu_{(1)} &= (1, -\sqrt{\tfrac{\Lambda}{3}}x^1, -\sqrt{\tfrac{\Lambda}{3}}x^2, -\sqrt{\tfrac{\Lambda}{3}}x^3) \\
\xi^\mu_{(2)} &= (0, 1, 0, 0) \\
\xi^\mu_{(3)} &= (0, 0, 1, 0) \\
\xi^\mu_{(4)} &= (0, 0, 0, 1) \\
\xi^\mu_{(5)} &= (0, x^2, -x^1, 0) \\
\xi^\mu_{(6)} &= (0, x^3, 0, -x^1) \\
\xi^\mu_{(7)} &= (0, 0, x^3, -x^2) \\
\xi^\mu_{(8)} &= (x^1, \tfrac{1}{2}\sqrt{\tfrac{\Lambda}{3}}[(x^1)^2 + (x^2)^2 + (x^3)^2 - \tfrac{3}{\Lambda}e^{2\sqrt{\frac{\Lambda}{3}}t}] - \sqrt{\tfrac{\Lambda}{3}}(x^1)^2, -\sqrt{\tfrac{\Lambda}{3}}x^1x^2, -\sqrt{\tfrac{\Lambda}{3}}x^1x^3) \\
\xi^\mu_{(9)} &= (x^2, -\sqrt{\tfrac{\Lambda}{3}}x^2x^1, \tfrac{1}{2}\sqrt{\tfrac{\Lambda}{3}}[(x^1)^2 + (x^2)^2 + (x^3)^2 - \tfrac{3}{\Lambda}e^{2\sqrt{\frac{\Lambda}{3}}t}] - \sqrt{\tfrac{\Lambda}{3}}(x^2)^2, -\sqrt{\tfrac{\Lambda}{3}}x^2x^3) \\
\xi^\mu_{(10)} &= (x^3, -\sqrt{\tfrac{\Lambda}{3}}x^3x^1, -\sqrt{\tfrac{\Lambda}{3}}x^3x^2, \tfrac{1}{2}\sqrt{\tfrac{\Lambda}{3}}[(x^1)^2 + (x^2)^2 + (x^3)^2 - \tfrac{3}{\Lambda}e^{2\sqrt{\frac{\Lambda}{3}}t}] - \sqrt{\tfrac{\Lambda}{3}}(x^3)^2)
\end{aligned} \tag{B.2}$$

In order to make references to Killing vectors easier, in equation (B.2) the notation $\xi^\mu_{(n)}$ signifies that the vector is the $n^{th}$ Killing vector. Let's start by looking at the first Killing vector, $\xi^\mu_{(1)}$.

$$\xi^{\mu}_{(1)} = (1, -\sqrt{\tfrac{\Lambda}{3}}x^1, -\sqrt{\tfrac{\Lambda}{3}}x^2, -\sqrt{\tfrac{\Lambda}{3}}x^3)$$
$$\xi_{\mu(1)} = (-1, -a^2\sqrt{\tfrac{\Lambda}{3}}x^1, -a^2\sqrt{\tfrac{\Lambda}{3}}x^2, -a^2\sqrt{\tfrac{\Lambda}{3}}x^3) \tag{B.3}$$

There is no time dependence in this vector, so (B.1a) is satisfied. Let's check (B.1b).

$$\begin{aligned} \partial_1\xi_0 &= 0 \\ \partial_0\xi_1 &= -2a\dot{a}\sqrt{\tfrac{\Lambda}{3}}x^1 \\ 2(\tfrac{\dot{a}}{a})\xi_1 &= 2(\tfrac{\dot{a}}{a})(-a^2\sqrt{\tfrac{\Lambda}{3}}x^1) = -2a\dot{a}\sqrt{\tfrac{\Lambda}{3}}x^1 \end{aligned} \tag{B.4}$$

So, $\partial_0\xi_1 - 2(\frac{\dot{a}}{a})\xi_1 = 0$, and (B.1b) is satisfied. We can see that (B.1c) and (B.1d) are of the exactly the same form as (B.1b), and so will yield the same result. Let's now check equation (B.1e).

$$\begin{aligned} \partial_1\xi_1 &= -a^2\sqrt{\tfrac{\Lambda}{3}} = -e^{2\sqrt{\frac{\Lambda}{3}}t}\sqrt{\tfrac{\Lambda}{3}} \\ a\dot{a}\xi_0 &= -a\dot{a} = -e^{2\sqrt{\frac{\Lambda}{3}}t}\sqrt{\tfrac{\Lambda}{3}} \end{aligned} \tag{B.5}$$

So, $\partial_1\xi_1 - a\dot{a}\xi_0 = 0$, and (B.1e) is satisfied. We can see that (B.1f) and (B.1g) are of the exactly the same form as (B.1e), and so will yield the same result. Looking at the Killing vector $\xi^{\mu}_{(1)}$, we see that we only have $x^1$ terms in the $1^{st}$ component of the Killing vector, $x^2$ terms in the $2^{nd}$ component of the Killing vector, etc. Thus, The last three Killing equations are clearly satisfied for the $\xi^{\mu}_{(1)}$. Let's now look at the next Killing vector.

$$\begin{aligned} \xi^{\mu}_{(2)} &= (0, 1, 0, 0) \\ \xi_{\mu(2)} &= (0, a^2, 0, 0) \end{aligned} \tag{B.6}$$

Since there is no time-dependence in the first component, equation (B.1a) is satisfied. Let's check equation (B.1b).

$$\begin{aligned} \partial_1\xi_0 &= 0 \\ \partial_0\xi_1 &= 2a\dot{a} \\ 2(\tfrac{\dot{a}}{a})\xi_1 &= 2(\tfrac{\dot{a}}{a})a^2 = 2a\dot{a} \end{aligned} \tag{B.7}$$

So, $\partial_0\xi_1 - 2(\frac{\dot{a}}{a})\xi_1$, and (B.1b) is satisfied. All other Killing equations are comprised of either $\xi_{0(2)} = \xi_{2(2)} = \xi_{3(2)} = 0$, or spatial derivatives of $\xi_{1(2)}$, and are thus all zero and all satisfied. By inspection we can see that the next two Killing vectors, $\xi_{\mu(3)}$ and $\xi_{\mu(4)}$, follow the same form as $\xi_{\mu(2)}$ and are also Killing vectors. Let's now check the next Killing vector $\xi^{\mu}_{(5)}$.

$$
\begin{aligned}
\xi^{\mu}_{(2)} &= (0, x^2, -x^1, 0) \\
\xi_{\mu(2)} &= (0, a^2x^2, -a^2x^1, 0)
\end{aligned}
\tag{B.8}
$$

There is no time-dependence in the first component so equation (B.1a) is satisfied. Let's check equation (B.1b).

$$
\begin{aligned}
\partial_1\xi_0 &= 0 \\
\partial_0\xi_1 &= 2a\dot{a}x^2 \\
2(\tfrac{\dot{a}}{a})\xi_1 &= 2(\tfrac{\dot{a}}{a})a^2x^2 = 2a\dot{a}x^2
\end{aligned}
\tag{B.9}
$$

So, $\partial_0\xi_1 - 2(\frac{\dot{a}}{a})\xi_1 = 0$ and equation (B.1b) is satisfied. Equation (B.1c) is of the same form as above, and equation (B.1d) includes only $\xi_{0(5)} = \xi_{3(5)} = 0$, and thus the Killing equations are satisfied. Equations (B.1e), (B.1f), (B.1g) include $\partial_j\xi_j$ which will always be zero given the form of the Killing vector, and $\xi_{0(5)} = 0$. So all three of these Killing equations are satisfied. Let's check equation (B.1h).

$$
\begin{aligned}
\partial_1\xi_2 &= -a^2 \\
\partial_2\xi_1 &= a^2
\end{aligned}
\tag{B.10}
$$

So, $\partial_1\xi_2 - \partial_2\xi_1 = 0$ and equation (B.1h) is satisfied. The last two Killing equations involve $\xi_{3(5)} = 0$ and $x^3$ derivatives of $\xi_{1(5)}$ and $\xi_{2(5)}$, which are clearly zero. Thus we can say that $\xi_{\mu(5)}$ is a Killing vector. We can also note that the next two Killing vectors $\xi_{\mu(6)}$ and $\xi_{\mu(7)}$ are of the same form as above, and will also be Killing vectors. We now just have the last three Killing vectors to check. We'll start with $\xi_{\mu(8)}$

$$
\begin{aligned}
\xi^{\mu}_{(8)} &= (x^1, \tfrac{1}{2}\sqrt{\tfrac{\Lambda}{3}}[(x^1)^2 + (x^2)^2 + (x^3)^2 - \tfrac{3}{\Lambda}e^{2\sqrt{\frac{\Lambda}{3}}t}] - \sqrt{\tfrac{\Lambda}{3}}(x^1)^2, -\sqrt{\tfrac{\Lambda}{3}}x^1x^2, -\sqrt{\tfrac{\Lambda}{3}}x^1x^3) \\
\xi_{\mu(8)} &= (-x^1, a^2\tfrac{1}{2}\sqrt{\tfrac{\Lambda}{3}}[(x^1)^2 + (x^2)^2 + (x^3)^2 - \tfrac{3}{\Lambda}e^{2\sqrt{\frac{\Lambda}{3}}t}] - \sqrt{\tfrac{\Lambda}{3}}(x^1)^2, -a^2\sqrt{\tfrac{\Lambda}{3}}x^1x^2, -a^2\sqrt{\tfrac{\Lambda}{3}}x^1x^3)
\end{aligned}
\tag{B.11}
$$

There is no time-dependence in the first component so equation (B.1a) is satisfied. Let's now check equation (B.1b).

$$
\begin{aligned}
\partial_1\xi_0 &= -1 \\
\partial_0\xi_1 &= 2a\dot{a}\tfrac{1}{2}\sqrt{\tfrac{\Lambda}{3}}((x^1)^2+(x^2)^2+(x^3)^2) - 2a\dot{a}\tfrac{1}{2}\sqrt{\tfrac{\Lambda}{3}}\tfrac{3}{\Lambda}e^{-2\sqrt{\frac{\Lambda}{3}}t}+ \\
&\quad 2\sqrt{\tfrac{\Lambda}{3}}a^2\tfrac{1}{2}\sqrt{\tfrac{\Lambda}{3}}\tfrac{3}{\Lambda}e^{-2\sqrt{\frac{\Lambda}{3}}t} - 2a\dot{a}\sqrt{\tfrac{\Lambda}{3}}(x^1)^2 \\
&= a\dot{a}\sqrt{\tfrac{\Lambda}{3}}((x^1)^2+(x^2)^2+(x^3)^2) - a\dot{a}\sqrt{\tfrac{\Lambda}{3}}\tfrac{3}{\Lambda}e^{-2\sqrt{\frac{\Lambda}{3}}t}+ \\
&\quad a^2e^{-2\sqrt{\frac{\Lambda}{3}}t} - 2a\dot{a}\sqrt{\tfrac{\Lambda}{3}}(x^1)^2 \\
2(\tfrac{\dot{a}}{a})\xi_1 &= 2a\dot{a}(\tfrac{1}{2}\sqrt{\tfrac{\Lambda}{3}}((x^1)^2+(x^2)^2+(x^3)^2 - \tfrac{3}{\Lambda}e^{-2\sqrt{\frac{\Lambda}{3}}t}) - \sqrt{\tfrac{\Lambda}{3}}(x^1)^2) \\
&= a\dot{a}\sqrt{\tfrac{\Lambda}{3}}((x^1)^2+(x^2)^2+(x^3)^2) - a\dot{a}\sqrt{\tfrac{\Lambda}{3}}\tfrac{3}{\Lambda}e^{-2\sqrt{\frac{\Lambda}{3}}t} - 2a\dot{a}\sqrt{\tfrac{\Lambda}{3}}(x^1)^2
\end{aligned} \tag{B.12}
$$

Combining these terms gives that:

$$
\begin{aligned}
\partial_1\xi_0 + \partial_0\xi_1 - 2(\tfrac{\dot{a}}{a})\xi_1 &= -1 + a\dot{a}\sqrt{\tfrac{\Lambda}{3}}((x^1)^2+(x^2)^2+(x^3)^2) - a\dot{a}\sqrt{\tfrac{\Lambda}{3}}\tfrac{3}{\Lambda}e^{-2\sqrt{\frac{\Lambda}{3}}t}+ \\
&\quad a^2e^{-2\sqrt{\frac{\Lambda}{3}}t} - 2a\dot{a}\sqrt{\tfrac{\Lambda}{3}}(x^1)^2 - a\dot{a}\sqrt{\tfrac{\Lambda}{3}}((x^1)^2+(x^2)^2+(x^3)^2)+ \\
&\quad a\dot{a}\sqrt{\tfrac{\Lambda}{3}}\tfrac{3}{\Lambda}e^{-2\sqrt{\frac{\Lambda}{3}}t} + 2a\dot{a}\sqrt{\tfrac{\Lambda}{3}}(x^1)^2 \\
&= -1 + a^2e^{-2\sqrt{\frac{\Lambda}{3}}t} \\
&= -1 + e^{2\sqrt{\frac{\Lambda}{3}}t}e^{-2\sqrt{\frac{\Lambda}{3}}t} \\
&= -1 + 1 \\
&= 0
\end{aligned} \tag{B.13}
$$

So, equation (B.1b) is satisfied. Let's now check equation (B.1c).

$$
\begin{aligned}
\partial_2\xi_0 &= 0 \\
\partial_0\xi_2 &= -2a\dot{a}\sqrt{\tfrac{\Lambda}{3}}x^1x^2 \\
2(\tfrac{\dot{a}}{a})\xi_2 &= 2(\tfrac{\dot{a}}{a})(-a^2\sqrt{\tfrac{\Lambda}{3}}x^1x^2 \\
&= -2a\dot{a}\sqrt{\tfrac{\Lambda}{3}}x^1x^2
\end{aligned} \tag{B.14}
$$

We see that $\partial_2\xi_0 + \partial_0\xi_2 - 2(\frac{\dot{a}}{a})\xi_2 = 0$, and thus equation (B.1b) is satisfied. Equation (B.1d) has the same form as above, and so $\xi^{\mu}_{(8)}$ will satisfy this equation as well. Let's now check equation (B.1e).

$$
\begin{aligned}
\partial_1\xi_1 &= 2a^2\tfrac{1}{2}\sqrt{\tfrac{\Lambda}{3}}x^1 - 2a^2\sqrt{\tfrac{\Lambda}{3}}x^1 \\
&= -a^2\sqrt{\tfrac{\Lambda}{3}}x^1 \\
&= -e^{2\sqrt{\frac{\Lambda}{3}}t}\sqrt{\tfrac{\Lambda}{3}}x^1 \\
a\dot{a}\xi_0 &= -x^1 a\dot{a} \\
&= -x^1\sqrt{\tfrac{\Lambda}{3}}e^{2\sqrt{\frac{\Lambda}{3}}t}
\end{aligned}
\tag{B.15}
$$

So we see that $\partial_1\xi_1 - a\dot{a}\xi_0 = 0$, and that equation (B.1e) is satisfied. Let's now go ahead and check equation (B.1f).

$$
\begin{aligned}
\partial_2\xi_2 &= -a^2\sqrt{\tfrac{\Lambda}{3}}x^1 \\
&= -e^{2\sqrt{\frac{\Lambda}{3}}t}\sqrt{\tfrac{\Lambda}{3}}x^1
\end{aligned}
\tag{B.16}
$$

We already showed above that $a\dot{a}\xi_0 = -x^1\sqrt{\frac{\Lambda}{3}}e^{2\sqrt{\frac{\Lambda}{3}}t}$ and so we can see that $\partial_2\xi_2 - a\dot{a}\xi_0 = 0$, and thus (B.1f) is satisfied. Equation (B.1g) is of the same form as (B.1f) and so it to is satisfied by this Killing vector. Now let's check equation (B.1h).

$$
\begin{aligned}
\partial_1\xi_2 &= -a^2\sqrt{\tfrac{\Lambda}{3}}x^2 \\
\partial_2\xi_1 &= 2a^2\tfrac{1}{2}\sqrt{\tfrac{\Lambda}{3}}x^2 \\
&= a^2\sqrt{\tfrac{\Lambda}{3}}x^2
\end{aligned}
\tag{B.17}
$$

We see that $\partial_1\xi_2 + \partial_2\xi_1 = 0$ and so equation (B.1h) is satisfied. We can see that equations (B.1i) and (B.1j) are of the same form as (B.1h), and so the Killing vector will satisfy these equations as well. So, $\xi^\mu_{(8)}$ is a Killing vector. Finally, we can see that the Killing vectors $\xi^\mu_{(9)}$ and $\xi^\mu_{(10)}$ are of the same form as $\xi^\mu_{(8)}$, and so they too are Killing vectors. We have now shown that our proposed set of ten Killing vectors for de Sitter space do in fact meet the requirements.

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