Theoretical mathematical model of fishery economics

Andre Garron
Colby College

Follow this and additional works at: http://digitalcommons.colby.edu/honorstheses

Part of the Economics Commons

Colby College theses are protected by copyright. They may be viewed or downloaded from this site for the purposes of research and scholarship. Reproduction or distribution for commercial purposes is prohibited without written permission of the author.

Recommended Citation
http://digitalcommons.colby.edu/honorstheses/94
A Theoretical Mathematical Model of Fishery Economics

Andre Garron
Advisor: Kashif Mansori, PhD
Second Reader: Thomas Berger, PhD
The sea hath fish for every man.

~William Camden, Remains Concerning Britain
Acknowledgements

There were many people involved in this project. I would first like to thank my two advisors, Kashif Mansori and Thomas Berger, for their large contributions of time, energy, and resources to this project. I would also like to thank George Welsh, Jan Holly, Thomas Tietenberg, Pamela Lefever, Timothy Lefever, and Sarah Kaminshine for their time and assistance.
Introduction

Human existence has always required the consumption of renewable resources. Although man may not live on bread alone, without food he cannot survive long. When the fields die, the farmers move. When the herd migrates, the tribe follows.

In a lucid moment betraying its pop-culture roots, *The Matrix* outlined the problem society faces now:

*Every mammal on this planet instinctively develops a natural equilibrium with the surrounding environment. But you humans do not. You move to an area and you multiply and multiply until every natural resource is consumed and the only way you can survive is to spread to another area.*

The world population recently topped 6 billion people. Each person has the same need to eat, breathe, and drink.

Unfortunately the world’s limited resources imply a carrying capacity for the human race. As we approach this critical mass, the strain on our resources becomes increasingly severe.

We have already begun to realize such disasters. Many of the world’s fisheries have been destroyed, depleted to near nonexistence, or critically overfished. In some cases they only held on because the price of catching the final fish outweighed the potential revenue from catching it.

This paper is an attempt to increase our understanding of the interaction between the economic motivations and biological forces at work in fisheries so that we may better preserve current stocks and help prevent future ecological devastation. More specifically, I seek to understand the decisions of fishermen in response to the price and stock of fish.

The basis for my work is “Backward Boycotts: Demand Management and Fishery Conservation”, by J. Samuel Barkin and Kashif Mansori, who argue that the supply curve for a fishery may downward sloping or be S-shaped. This paper is divided into the
following sections: I. Basic Fishery Economics; II. Relevant Literature; III. Derivation of the Model: Assumptions; IV. Proof that the Supply Curve is Backwards Bending; V. Derivation of the Model: Deriving a System of Equations; VI. Derivation of Parameters for System of Equations; VII. The Resulting Model; VIII. Dynamic Shifts; IX. Conclusion; X. Maple Appendix.¹

¹ The Maple files have been converted to text file format and most outputs have been deleted for simplicity.
I. Basic Fishery Economics

In the past half century, consumption of fish has nearly doubled. The Food and Agriculture Organisation (FAO) reported that 95m tons of fish were landed in 2000, seemingly the highest catch ever. However this number may be inflated, thanks to inaccurate reporting. It is suspected that the yearly catch has declined since the 1980s, a result of almost 75% of the world's fisheries being depleted.²

Fisheries are interactive resources: they are subject to harvesting, which depletes their stock, and biological factors, by which nature generally attempts to increase the stock. Left to nature's influences, fishery populations approach a biological carrying capacity over time. Generally this is modeled as a logistic curve where the rate of fish growth reaches a maximum somewhere in between zero stock and maximum stock. The stock's highest rate of growth is also its maximum sustainable yield. If fishermen remove exactly as many fish as are bred into the environment each year, then the total catch will be the largest that can be continued perpetually. A stock size above or below the MSY decreases the rate of growth. There may even be a minimum quantity of stock below which the growth rate is negative, perhaps because the population's density is low enough to keep fish from the actual act of reproduction. This is the basis of the Schaefer model of the fishery: there is a relationship between the stock of fish and the growth rate of the stock.

This is shown in Figure 1.1, where \( S_{\text{min}} \) represents the minimum sustainable stock, \( S^* \) is the stock at the MSY, \( S_{\text{max}} \) is the maximum sustainable stock.³

² "Special Fishing Report," pg. 19
³ For further discussion of fishery economics, see Tom Tietenberg, Environmental Economics & Policy, pg.s 221-239
Fishermen have conflicting motives when deciding how much to fish. On one hand they want to maximize the discounted sum of the present and future net benefits from a fishery. Barring unusual circumstances, such as an infinite discount rate, their general inclination is to fish at the MSY, should price be held constant for changes in the quantity supplied. This would produce a dynamic equilibrium.

However, in an open-access fishery, like in any other industry, economic profits encourage competition. Since a fisherman cannot control the actions of the other fishermen, he cannot ensure a future stock of fish, he develops an incentive to fish more in the present. Instead of maximizing his net benefits from fishing, he drives rent down to zero where all fishermen realize economic profits. This is a manifestation of the problem of the commons. The individual has an incentive to engage in behavior that is harmful to society as a whole. In the case of fishing, the incentive to overfish can lead to lower future fish harvests or permanently destroy the fishery.

We can see this conflict in Figure 1.2. If the marginal cost is constant as fishing effort increases then the Total Cost function is a straight line. The Total Benefit curve
declines as effort increases because this model assumes sustainability into the future and constancy in effort. Thus, when effort has driven the catch passed the MSY, future harvests will decrease, decreasing the present value of those harvests. Clearly, fishermen have the greatest surplus at \( E_c \), where their effort matches the slope of the Total Cost curve to the slope of the Total Benefit curve. However, as described, open-access competition will push effort to where the TB and TC curves intersect, at \( E_c \).

Thus, static efficiency and dynamic efficiency conflict in a competitive fishery.

Figure 1.2

There are two types of economic externalities created here. The contemporaneous externality, the overcommitment of resources to fishing, affects the current generation. That is, resources being used for fishing could benefit society more if used elsewhere. The intergenerational externality is the effect on future generations of a lower stock due to overfishing. This includes smaller future harvests and, in general, a lower discounted present value for use of the fishery.

\(^4\) Here I am assuming that the discount rate of future goods is not large enough to interfere.
Possible responses to the problems of the fishery include aquaculture, fish farming, regulating equipment, taxes, individual transferable quotas (ITQs), and boycotts and other attempts to change consumer demand. Aquaculture, or fish farming, is the practice of making the resource privately owned. Fish farmers generally work a stock that is artificially enclosed by fences. Since the incentive to overfish only occurs when market entry eliminates the gains from preserving the stock, a privately owned fishery should generally seek to operate at an efficient rate of harvest. Equipment regulation (such as forbidding the use of certain technologies) increases the cost of fishing, which reduces the catch, but clearly works to also decrease efficiency. Attempts to lower consumer demand assume that if the demand curve drops, then the equilibrium price will drop, and fewer fish will be supplied. ITQs work by encouraging fishermen to remove their quota of fish as efficiently as possible. The quotas are set so as to maintain or increase the stock of fish.5

Each regulation makes certain assumptions about fishery economics. For instance, regulations and practices that affect the demand curve, such as aquaculture or boycotts, assume an upward sloping supply curve. My model addresses this particular assumption by assuming a backwards bending, or downward sloping, supply curve. In short, the model can evaluate the effects of different economic assumptions on the effectiveness of regulations.

6 For a review of mathematics and economics associated with renewable resource analysis, see Frederick M. Peterson and Anthony C. Fisher, "The Exploitation of Extractive Resources: A Survey."
Relevant Literature

"Backward Boycotts: Demand Management and Fishery Conservation" by J. Samuel Barkin and Kashif Mansori serves as my point of departure. It concludes that the supply curve for Northeastern fisheries "show characteristics of a backward-bending supply curve." Barkin and Mansori analyzed the elasticity of supply and demand using data from the New England Fishery. The elasticity of demand was consistently normal (negative) and significant at the 99% level. Their data suggested an elasticity of -2.6. However, the elasticity of supply was found to be -1, suggesting that the supply curve also slopes downward, instead of upwards as is conventionally assumed.

They explained these results by suggesting that fishermen have basic financial needs that must be met with their fishing income. When the price drops significantly, instead of exiting the market or decreasing the quantity supplied, fishermen actually fish more so as to meet their financial obligations.

The same phenomenon has been noted in labor markets in general. When wages drop far enough, workers actually work more hours so as to meet their minimum financial needs.

Fishermen have a number of possible motivations for such behavior. First, if boats are bought on borrowed money then fishermen need some level of revenue to make the interest payments. Next, barriers to exit and re-entry encourage fishermen to avoid leaving the industry. Even the cultural attachment to fishing is an incentive to remain in the industry.

Figure 2.1 is a graphical representation of a conventional demand shock and Figure 2.2 shows the same shock with a backward bending supply curve. As you can see from Figure 2.2, demand policies actually encourage further overfishing if the demand curve is backward bending.\(^7\)

\(^7\) Barkin and Mansori, "Backward Boycotts: Demand Management and Fishery Conservation," pg.s 30-41.

\(^8\) Ibid.
The result is that attempts to control demand may be an ineffective, and potentially harmful, means of influencing a fishery. These attempts include anything that seeks to reduce consumer demand and thereby reduce the quantity of fish supplied, such as consumer boycotts or aquaculture.9

Conventional demand policies assume a standard upward sloping supply curve. Thus, when a policy encourages people to eat less fish, the demand curve should drop. However, if the supply curve is S-shaped, as in Figure 2.2, decreased demand increases the quantity supplied, and the policy is accomplishes the opposite of what was intended. On the other hand, supply-side policies, such as ITQs, will be effective.10

---

9 I refer to such actions as “demand policies.”
10 Ibid.
Figure 2.1

Price

Figure 2.2

Price

Quantity

Quantity
One interesting point arises when this logic is applied to the Schaefer model. A fisherman will only increase effort when the marginal cost of the effort is still below the marginal benefit of effort. For the fisherman to increase effort when revenue per fish falls, he would need to be originally using an inefficiently low amount of effort. In theory, he already could have been making more profit by using more effort, and therefore should not have needed a price drop to motivate him to increase his effort.

I reconcile this by considering that the fisherman is interested in consuming more than one activity. While work provides revenue, he will only work until the marginal utility from income equals the marginal utility from other activities, like leisure. We can suppose that the risk of being forced out of the industry from a price drop creates an unusual circumstance that increases the marginal utility of income.

In my model price is endogenous to reflect the possibility that the amount of fishing affects the price. Since Barkin and Mansori surveyed real-world data, they simply observed price as it responded to a variety of factors.

Colin Clark's book, *Mathematical Bioeconomics*, is the most comprehensive use of differential equations and advanced non-econometric mathematics to study renewable resources. The beginning of his work is a mathematical discussion of the economics discussed in I. He then considers several different dynamic fishery models.

Only in one case does he consider a backward bending supply curve. He motivates it by considering the biological overfishing that occurs when the harvest rate surpasses the maximum sustainable yield. In this case, not counting the first period of increased effort, any sustained harvest beyond the MSY must be lower than the MSY. Thus, as price increases, and effort increases correspondingly, the supply curve bends backwards once the yield passes the MSY. Note that Clark's model assumes a constant level of fishing effort into the future. Otherwise harvests above the MSY could be obtained for several periods by simply depleting the stock of fish.

However, in my model I do not assume a sustainable harvest is created at each price. This allows me to examine factors that can cause the stock to be driven to zero over time as the price is kept from varying. Furthermore, the backward bending effect in my supply curve is motivated by the economic choices of fishermen, not the biological

---

constraints of fishery growth. Finally, Clark’s model of the harvest does not respond to price, but only to the stock of fish. My model considers the effect of both the stock and the price on the quantity supplied.

The Ricker Salmon Model is another dynamic analysis of fisheries. It models the rate of growth of a stock of salmon in the absence of harvesting and the dynamic path to or away from equilibrium. The model determines the rate of growth through variables including the adult population size, the rate of growth of juveniles (which acts as a measure of population density, creating effects similar to my use of a logistic equation), and the mortality of the juveniles. However, Greenwell and Ng’s discussion admits that the model is deterministic and does not conveniently include the effect of random variables, such as climate.

Interestingly, Greenwell and Ng found that the model exhibits chaotic behavior, depending on the value assigned to the constant of proportionality between the number of recruits and the number of adults in the next generation. Further, by changing the values assigned to their proportionality constants, they could manipulate the system from a stable to an unstable equilibrium. This corresponds well with my model, which shows the existence of a stable equilibrium level of stock to depend on the intrinsic rate of growth of the fish stock.

Greenwell and Ng also provide an interesting perspective on the incentives of fishermen. They assume that fishermen do not want a sustainable catch indefinitely, nor that fishermen want to maximize the present net value of all use of the fishery. And, in truth, these interests certainly are closer to interests of society than fishermen. Instead, fishermen seek to maximize the total catch for a finite time period, perhaps the remainder of his life or career. Thus the fisherman has no economic incentive to leave a viable fishery. However, their analysis of this motive is not useful to me since they do not allow for variations in price from changes in supply.

James A. Brander and M. Scott Taylor did a dynamic analysis similar to mine for resource use on Easter Island. They attempted to explain the disappearance of the civilization that erected the island’s giant statues. They determined that, given a slow growing resource, the human population would overshoot equilibrium with the island’s

12 Greenwell and Ng, "The Ricker Salmon Model."
natural resources and collapse. This is consistent with my model, where we will see that the intrinsic growth rate of fish and the rate of harvest can affect the model nearly identically.13

Tracy Lewis notes, as Greenwell and Ng explained, that fisheries are not deterministic in their behavior. Instead, they have a variety of unpredictable economic and biological and climatic influences. He forms a system for decision-making in fishery regulation given this uncertainty and finds that optimal strategies are based on the level of uncertainty and the regulator’s preference for risk-taking.14 I can only note that this discussion of uncertainty is prevalent among researchers and offer this caveat to my work: a model is always a simplification of reality.

Gordon Munro explained competition in a fishery as a prisoner’s dilemma. Each fisherman has an incentive to restrain fishing so the fishery is not destroyed. However, this would require personally curtailing one’s catch. Since no fisherman has the ability to control the behavior of other fishermen, he will fish to the point that Total Cost and Total Revenue become equal, depriving both fishermen and society of any economic surplus, as discussed previously.15

Charles Mason and Stephen Polasky consider the effect of changes in the number of firms in a commonly shared resource: “Increasing industry size raises costs but lowers prices; thus it has ambiguous welfare effects.” They determined that the optimal number of firms changes as the total stock changes.16 In my model the number of firms is endogenous. However, instead of changing to reflect an optimal number, it changes in response to economic incentives; as profits increase fishermen enter the industry.

In a study of the collapse of cod stocks in the Atlantic outside of Canada, Ransom A. Myers, Jeffrey A. Hutchings, and Nicholas J. Barrowman tested whether poor recruitment contributed. Their results suggested that it did not and that overfishing was the outstanding culprit. This lends credibility to my model, which, although lacking a stochastic variable for unpredictable movements in biological or economic conditions, looks towards the total effort put into fishing to explain changes in stock. If low

13 Brander and Taylor, “The Simple Economics of Easter Island.”
15 Munro “Approaches to the Economics of the Management of High Seas Fishery Resources.”
16 Mason and Polasky, “The Optimal Number of Firms in the Commons.”
recruitment were important in the collapse of these stocks, then my model, which exogenously mandates the recruitment, would be less useful.\textsuperscript{17}

\textsuperscript{17} Myers, Hutchings, and Barrowman, "Why do Fish Stocks Collapse?"
III.

Derivation of the Model:

Assumptions

Almost all of my work was completed on *Maple V*, a mathematical software program. The Appendix includes the *Maple* documents I used to derive the system of equations and analyze the phase plane. It is attached only for those who wish to fully investigate the mathematical logic I followed.

I begin my assumptions with the Schaeffer model of fisheries:

\[
\frac{dx(t)}{dt} = F(x(t)) - h(x(t))
\]

\[
F(x(t)) = r \cdot x(t) \cdot (1 - \frac{x(t)}{k})
\]

Here the stock of fish at a time \( t \) is represented by \( x(t) \). The change in the stock is a function of the natural growth rate of the stock, \( F(x(t)) \), minus the rate at which the stock is harvested, \( h(x(t)) \). \( F(x(t)) \) is a function of a population limit, \( k \), the intrinsic rate of growth of the stock, \( r \), and the stock size at that time. The natural rate of growth of the stock is therefore logistic.

I chose a linear demand curve for simplicity:

\[
p(t) = a - b \cdot Q(t)
\]

Here price at time \( t \), \( p(t) \), is a function of the quantity of fish demanded at \( t \), \( Q(t) \), and the linear parameters \( a \) and \( b \). The demand curve is assumed to be linear for simplification.

If I assume that the quantity demanded always equals the quantity supplied, then I can further specify \( Q(t) \) as the quantity of fish supplied at time \( t \). Thus, I set it equal to the number of firms/fishermen in the industry, \( n \), multiplied by the quantity supplied per fisherman, \( q(t) \). This assumes homogeneity among the motivations and choices of fishermen.
\( Q(t) = n(t) \cdot q(t) \)

I establish the number of firms, \( n \), as an extrinsic variable. Given economic profits, firms should enter the industry. Given losses, firms should exit. Thus, \( n \) is a function of the profits of fishermen, \( Y(t) \).

\[ n(t) = j1 + j2 \cdot Y(t) \]

I can still treat \( n \) as an endogenous variable by setting \( j2 \), what may be called the marginal propensity of firms to enter or exit, equal to 0. Then specification of \( j1 \) establishes the number of firms for the system.

My utility function was originally chosen for simplicity, and set to be \( U(t) = Y(t) \cdot L(t) \), a function of income and hours of leisure, \( L(t) \), enjoyed. I have since updated it to a Cobb-Douglas function:

\[ U(t) = cd \cdot Y(t)^{cy} \cdot L(t)^{cl} \]

I can still return to the original function by setting \( cd, cy \) and \( cl \) equal to 1. Naturally, if profit from fishing is zero, utility will be zero as well.

Use of the Cobb-Douglas function allows me to control the returns to scale of income and leisure. If \( cy \) and \( cl \) sum to 1 then there are constant returns to scale. If they sum to more than one then there are increasing returns to scale. If they sum to less than one then there are decreasing returns to scale. My notation is thus more flexible than the frequent notation of \( U = Y^Z \cdot L^{-z} \), which can only sum to 1.\(^{18} \)

Income is simply set equal to total costs minus total revenue:

\[
\begin{align*}
Y(t) &= TR(t) - TC(t) \\
TR(t) &= q(t) \cdot p(t) \\
TC(t) &= w(t) \cdot f + I
\end{align*}
\]
In the total cost equation, \( w \) is the hours devoted to work, or fishing, \( f \) is the marginal costs of fishing per hour, including fuel, and \( I \) represents sunk costs such as investment in capital. Although \( I \) is entered as a constant in this version of the model, it may be useful later to let it change in response to interest rates.

Here it may be helpful to note that, although the equations are expressed as instantaneous rates of change, I run the model by setting the period of time \( t \) equal to a week. For example, \( w(t) \) does not refer to instantaneous consumption of work, but instead to work consumed in period \( t \). Likewise, \( \frac{dx(t)}{dt} \) refers to the change in the stock of fish in one week, as opposed to the instantaneous rate of change of the stock. So although I use the notation of differential equations, the model should be interpreted as difference equations. Since I have chosen a week as my timeframe, the hours of leisure consumed per week becomes:

\[
L(t) = 168 - w(t)\]

I specify the amount of fish a fisherman can catch according to the Schaeffer model. It is dependent on the stock of fish in the sea \( x(t) \), the hours worked \( w \), and the efficiency of fishermen, \( a \). The stock of fish is relevant because, as the population density drops, it becomes more difficult to catch fish. Furthermore, \( a \) can be used as a proxy of technology.\(^{20}\) This specification of \( q(t) \) can then be substituted into \( Y(t) \).

\[
q(t) = a \cdot x(t) \cdot w(t) \cdot u
\]
\[
Y(t) = (a \cdot x(t) \cdot w(t) \cdot u \cdot p(t) - w(t) \cdot f - I)
\]

\(^{18}\) For further discussion of Cobb-Douglas functions, see Brown, “Cobb-Douglas Function” and Pearce, ed., “Cobb-Douglas production function.”

\(^{19}\) Another interesting function would be to set \( L(t) = 1 - w(t) \), so that leisure and work would be measured as percents of total time in a period.

\(^{20}\) An untested idea is to let \( a \) increase with time to model advances in technology. But, considering the short timeframe of my model, such changes are irrelevant.
The variable $u$ can be viewed as a parameter for analysis of season regulation. It is motivating a change in harvesting efficiency in response to shorter fishing seasons being enforced.

Table 3.1 specifies the units of each variable in my assumptions. It is necessary to know the units involved because I later specify value for variables in the model and some of these specifications are based on real world data.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(x)$</td>
<td>lbs of growth of the fish stock in one week</td>
</tr>
<tr>
<td>$h(x)$</td>
<td>lbs of fish harvested in one week</td>
</tr>
<tr>
<td>$r$</td>
<td>Intrinsic biological rate of growth of the fish stock; percent growth per week</td>
</tr>
<tr>
<td>$x$</td>
<td>lbs of fish in the fishery</td>
</tr>
<tr>
<td>$k$</td>
<td>Maximum sustainable stock of fish in lbs</td>
</tr>
<tr>
<td>$p$</td>
<td>$ Price of 1 lb of fish</td>
</tr>
<tr>
<td>$a$</td>
<td>Autonomous demand for fish</td>
</tr>
<tr>
<td>$b$</td>
<td>Marginal propensity to consume fish</td>
</tr>
<tr>
<td>$Q$</td>
<td>lbs of fish supplied by all fishermen in the industry</td>
</tr>
<tr>
<td>$n$</td>
<td>Number of fishermen/firms in the fishery</td>
</tr>
<tr>
<td>$q$</td>
<td>lbs of fish harvested by one fisherman in one period</td>
</tr>
<tr>
<td>$U$</td>
<td>Utility of fishermen</td>
</tr>
<tr>
<td>$Y$</td>
<td>$ Income</td>
</tr>
<tr>
<td>$L$</td>
<td>Hours of leisure consumed per week</td>
</tr>
<tr>
<td>$TR$</td>
<td>Total Revenue</td>
</tr>
<tr>
<td>$TC$</td>
<td>Total Costs</td>
</tr>
<tr>
<td>$w$</td>
<td>Hours worked per week</td>
</tr>
<tr>
<td>$f$</td>
<td>Marginal cost of 1 hour of fishing</td>
</tr>
<tr>
<td>$l$</td>
<td>$ Fixed costs of fishing per week</td>
</tr>
<tr>
<td>$g$</td>
<td>% of available stock caught by 1 fisherman in 1 hour</td>
</tr>
<tr>
<td>$u$</td>
<td>Scaling factor for productivity</td>
</tr>
<tr>
<td>cd</td>
<td>Efficiency parameter in Cobb-Douglas utility function</td>
</tr>
<tr>
<td>-----</td>
<td>-----------------------------------------------------</td>
</tr>
<tr>
<td>cl</td>
<td>Elasticity of utility with respect to leisure</td>
</tr>
<tr>
<td>cy</td>
<td>Elasticity of utility with respect to income</td>
</tr>
<tr>
<td>j1</td>
<td>Number of fishermen/firms at economic profits</td>
</tr>
<tr>
<td>j2</td>
<td>Propensity of fishermen to enter or leave industry; fishermen per dollar</td>
</tr>
</tbody>
</table>
IV.

Proof that the Supply Curve is Backwards Bending

Given my assumptions, an individual fisherman seeking to maximize utility will make decisions that create a backward bending supply curve. Thus, as the price of fish rises, *ceteris paribus*, a fisherman will actually work less.

What follows is a mathematical proof of this phenomenon:

Assumptions:

\[ U = cd \cdot Y^\alpha \cdot L^\beta \]
\[ Y = p \cdot \alpha \cdot w \cdot x - w \cdot f - I \]
\[ L = T - w \]

Proof:

\[
U = cd \cdot (p \cdot \alpha \cdot w \cdot x - w \cdot f - I)^\alpha \cdot (T-w)^\beta
\]
\[
dU = \frac{cd \cdot cy \cdot (p \cdot \alpha \cdot w \cdot x - w \cdot f - I)^\alpha \cdot (p \cdot \alpha \cdot x - f)}{p \cdot \alpha \cdot w \cdot x - w \cdot f - I} \cdot \frac{cd \cdot cl \cdot (p \cdot \alpha \cdot w \cdot x - w \cdot f - I)^\alpha \cdot (T-w)^\beta}{T-w} \]
\[
\frac{dU}{dw} = 0
\]

\[
let \quad \frac{dU}{dw} = 0
\]

\[
\frac{cd \cdot cy \cdot (p \cdot \alpha \cdot w \cdot x - w \cdot f - I)^\alpha \cdot (T-w)^\beta \cdot (p \cdot \alpha \cdot x - f)}{p \cdot \alpha \cdot w \cdot x - w \cdot f - I} = \frac{cd \cdot cl \cdot (p \cdot \alpha \cdot w \cdot x - w \cdot f - I)^\alpha \cdot (T-w)^\beta}{T-w}
\]

\[
cy \cdot (T-w) \cdot (p \cdot \alpha \cdot x - f) = cl \cdot (p \cdot \alpha \cdot w \cdot x - w \cdot f - I)
\]

\[
w = \frac{cy \cdot T \cdot f - cl \cdot I - cy \cdot T \cdot p \cdot \alpha \cdot x}{cy \cdot f + cl \cdot f - cy \cdot p \cdot \alpha \cdot x - cl \cdot p \cdot \alpha \cdot x} = \frac{cy \cdot T \cdot (f - p \cdot \alpha \cdot x) - cl \cdot I}{(cy + cl) \cdot (f - p \cdot \alpha \cdot x)}
\]

\[
w = \frac{cy \cdot T \cdot (f - p \cdot \alpha \cdot x) - cl \cdot I}{(cy + cl) \cdot (f - p \cdot \alpha \cdot x)}
\]

\[
w = \frac{cy \cdot T \cdot cl \cdot I}{(cy + cl) \cdot (f - p \cdot \alpha \cdot x)}
\]

These are all equations from my previous assumptions. However, I changed the time frame from one week, 168 hours, to the more flexible \( T \).
From the final equation we can see that any increase in price, $p$, will increase the second term in the function, causing the hours worked, $w$, to decrease. Thus, the supply curve bends backwards. Mathematically, this results from assuming a Cobb-Douglas utility function and positive fixed costs.
V.

Derivation of the Model:

Deriving a System of Equations

The steps in the derivation of this model produce extremely long and complicated formulas. As a result, I will describe the process and let the more intrepid follow it on the Maple attachments. I will then present the system of equations that resulted.

The first step is to substitute \( q(t) \) into the total revenue formula. Then this new total revenue formula and the total cost formula are substituted into the formula for income, \( Y(t) \). This formula is then substituted into \( n(t) \), which in turn is substituted into \( Q(t) \).

I then substitute my functions for income and leisure into the utility function. I take this new utility function, differentiate it with respect to \( w \), and then solve for \( w \). I substitute this into the most recent formula for \( q(t) \) which is then substituted it and the formula for \( w \) into the most recent formula for \( Q(t) \). I now have the terms necessary for the first equation, \( \frac{dx(t)}{dt} \).

To find the next equation I differentiated a \( Q(t) \) function with respect to \( t \) and solved for \( \frac{dp(t)}{dt} \). This produces the system of equations. These equations are expressed as Maple V expressed them:

\[
\begin{align*}
\frac{d}{dt} x(t) &= r x(t) \left( 1 - \frac{x(t)}{k} \right) - \left( \frac{g x(t) (16 + c g x(t) u p(t) - 16 + c y f + c l l)}{c y g x(t) u p(t) - c y f + c l g x(t) u p(t) - c l f} \right) \left( \frac{(16 + c y g x(t) u p(t) - 16 + c y f + c l l) f}{(16 + c y g x(t) u p(t) - 16 + c y f + c l l)} \right) g x(t) \\
(16 + c y g x(t) u p(t) - 16 + c y f + c l l) u) / (c y g x(t) u p(t) - c y f + c l g x(t) u p(t) - c l f)
\end{align*}
\]
\[
\frac{d}{dt} p(t) = -g \left( \frac{d}{dt} x(t) \right) u \left( a p(t) c t^2 - b j l c t^2 II - cy b j l c t II + 168 f c y b j l c t \right) \\
+ 168 f c y^2 b j l - b j l \left( \frac{168 f - 168 g x(t) u p(t) + II \text{ cy}}{cy + cl} \right) cl^2 II \\
- cy b j l \left( \frac{168 f - 168 g x(t) u p(t) + II \text{ cy}}{cy + cl} \right) cl II \\
+ 168 f c y^2 b j l \left( \frac{168 f - 168 g x(t) u p(t) + II \text{ cy}}{cy + cl} \right) cl \\
+ 168 f c y^2 b j l \left( \frac{168 f - 168 g x(t) u p(t) + II \text{ cy}}{cy + cl} \right) - 2 cy p(t)^2 cl + cy^2 a p(t) \\
- cy^2 p(t)^2 - p(t)^2 cl^2 + 2 cy a cl p(t) - 336 cy^2 b g x(t) u j l p(t) \\
- 336 cy b g x(t) u j l p(t) cl \\
- 336 cy^2 b g x(t) u j l \left( \frac{168 f - 168 g x(t) u p(t) + II \text{ cy}}{cy + cl} \right) \ p(t) \\
- 336 cy b g x(t) u j l \left( \frac{168 f - 168 g x(t) u p(t) + II \text{ cy}}{cy + cl} \right) \ p(t) cl \\
+ 28224 f c y^2 b g x(t) u D(j2) \left( \frac{168 f - 168 g x(t) u p(t) + II \text{ cy}}{cy + cl} \right) p(t) \\
- 168 cy b g x(t) u D(j2) \left( \frac{168 f - 168 g x(t) u p(t) + II \text{ cy}}{cy + cl} \right) cl II p(t) \\
- 28224 cy^2 b g^2 x(t)^2 u^2 D(j2) \left( \frac{168 f - 168 g x(t) u p(t) + II \text{ cy}}{cy + cl} \right) p(t)^2 \bigg) / \\
\left( 2 cy a cl g x(t) u - 4 cy g x(t) u p(t) cl \right) \\
- 168 cy^2 b g^2 x(t)^2 u^2 j l \left( \frac{168 f - 168 g x(t) u p(t) + II \text{ cy}}{cy + cl} \right) \\
- 168 cy^2 b g^2 x(t)^2 u^2 j l \\
- 168 cy b g^2 x(t)^2 u^2 D(j2) \left( \frac{168 f - 168 g x(t) u p(t) + II \text{ cy}}{cy + cl} \right) cl II \\
+ 28224 f c y^2 b g^2 x(t)^2 u^2 D(j2) \left( \frac{168 f - 168 g x(t) u p(t) + II \text{ cy}}{cy + cl} \right) p(t) \\
- 168 cy b g^2 x(t)^2 u^2 j l cl \\
- 168 cy b g^2 x(t)^2 u^2 j l \left( \frac{168 f - 168 g x(t) u p(t) + II \text{ cy}}{cy + cl} \right) cl \\
- 28224 cy^2 b g^3 x(t)^3 u^3 D(j2) \left( \frac{168 f - 168 g x(t) u p(t) + II \text{ cy}}{cy + cl} \right) p(t) \\
- 2 cy^2 g x(t) u p(t) + a g x(t) u c t^2 + cy^2 a g x(t) u - 2 g x(t) u p(t) cl^2 + f c l^2 \\
+ 2 f c y c l + f c y^2 \right) 
\]
VI.

Derivation of Parameters for System of Equations

Unfortunately, it is not possible to solve for meaningful equilibriums or phase movements with these equations as they are. In order to create a graphical phase plane, I need to numerically simulate the parameters in an actual fishery. Therefore, I need to pick values for the parameters that make sense when considered together. Otherwise the calibration would produce results that do not mimic reality. The more realistic I make the values I pick, the more useful the model will be.

I can set certain parameters almost arbitrarily because the actual values do not matter as much as the relations between the values or because they are only important relative to the analysis of the phase plane. For instance, the maximum sustainable population, $k$, was picked as 132,300,000 lbs of fish. The variable costs per hour, $f$, are set to 1 and the fixed costs per week, $I$, are set to 7, a fixed cost of 1 per day.

I decided to use 1/3 as the maximum possible growth in stock, the MSY. To determine the intrinsic rate of growth of the system, $r$, I entered this value into the logistic formula:

$$F(x(t)) = x(t) \cdot \frac{1}{3} = r \cdot x(t) \cdot (1 - \frac{x(t)}{k})$$

Since $r$ is equal to the growth rate at the MSY, and since the MSY occurs where $x(t) = \frac{k}{2}$, we can solve for $r$ as follows:

---

22 Despite the apparent specificity of $k$, it was based loosely on data on cod landings in *Fisheries of the United States, 2002* (Pritchard ed.) and is a rather significant underestimation of the actual maximum population sustainable in a cod fishery. However, this model's analysis applies to relative, not absolute movements in stock, making the discrepancy irrelevant.

23 This was based loosely on Atlantic cod data from *Status of Fishery Resources off the Northeastern United States for 1998* (Clark ed.). After completing my analysis with my data, I found Mayo and O'Brien's article "Atlantic Cod," which gives precise data for an Atlantic cod fishery. Future research may want to use this more realistic data as I have not.
\[ \frac{1}{3} = r \cdot \left( \frac{1}{2} \right) \]
\[ r = \frac{2}{3} \]

However, since my period of time is one week, and 1/3 stock growth refers to growth over a year, the r that I am looking for is 2/3 divided by 52 weeks. Thus, \( r = 0.01269 \).\(^{24}\)

For this first model I also set \( \alpha \) so that the maximum stock fishermen can remove in one year is equal to the MSY. That is to say, if fishermen consumed no leisure, and fished constantly, each could only catch 0.00038% of the fish in the ocean in one hour.

\[ \frac{0.01269 \text{ fish}}{\text{week}} \div 168 \frac{\text{hours}}{\text{week}} \div 100 \frac{\text{fishermen}}{\text{week}} = 0.00000038 \]

This approximation assumes 100 fishermen in the industry. Thus, I set the number of fishermen if there are no economic profits, \( j_1 \), equal to 100. The propensity of fishermen to enter and exit the industry, \( j_2 \), was set to 0.0001 by trial and error, so there are not impossibly large or irrelevantly small changes in the number of fishermen as profits change.

The demand parameters were implied through data of the price and quantity of cod in consecutive years. Table 6.1 displays the data that was used.

Table 6.1

<table>
<thead>
<tr>
<th></th>
<th>2001</th>
<th>2002</th>
</tr>
</thead>
<tbody>
<tr>
<td>weight*</td>
<td>value**</td>
<td>weight</td>
</tr>
<tr>
<td>Atlantic Cod</td>
<td>33,211</td>
<td>32,086</td>
</tr>
<tr>
<td>Harvest</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\* weight in 1000 lbs

\** value in $1000

In this and some other calculations I simply ignore the compounding effects of stock growth, which are marginal at best.
I set up the following simultaneous equations and solved for \( a \) and \( b \):

\[
\frac{32,086,000}{33,211,000} = a - b \cdot 33,211,000
\]

\[
\frac{30,715,000}{28,941,000} = a - b \cdot 28,941,000
\]

\[a = 1.7064\]
\[b = .0000002\]

In doing so I make a large assumption about this data. I have assumed that the change in price and quantity is a result of a shift in the supply curve. However, Joshua D. Angrist, Kathryn Graddy, and Guido W. Imbens have pointed out that identifying either curve "requires the presence of separate instruments that shift either demand or supply but not both."\(^{25}\) The data I used may reflect shifts in either or both curves, not just the supply curve as I would like.

Of the other variables, I initially set \( cd, cl, \) and \( cy \) equal to 1. Thus constructs the simplest utility function: \( U = Y \cdot L \).

Figure 6.1 shows the supply curve, which was previously shown to be backward bending, given these parameters. Since, for the chosen values of \( cl \) and \( cy \), income and leisure are weighted equally, fishermen will choose to work about half a week, depending on the price. The curve is asymptotic to \( w=84 \).

\(^{25}\) Angrist, Graddy, and Imbens, "The Interpretation of Instrumental Variables Estimators in Simultaneous Equations Models with an Application to the Demand for Fish," pg 500.
Figure 6.1

$ Price

0

1

84
VII.
The Resulting Model

Once I had substituted the parameter specifications into the system of equations I solved for equilibriums and nullclines. Equilibriums are price and stock levels at which the price and stock no longer wish to move. That is, at equilibrium \( \frac{dp(t)}{dt} \) and \( \frac{dx(t)}{dt} \) will effectively equal 0 or \( \infty \). Nullclines are points where either price or quantity has no pressure to move, but not both. Nullclines have at least one of the following characteristics: \( \frac{dp(t)}{dt} = 0 \), \( \frac{dx(t)}{dt} = 0 \), \( \frac{dp(t)}{dt} = \infty \), \( \frac{dx(t)}{dt} = \infty \). If two nullclines intersect, one with the characteristic that \( \frac{dp(t)}{dt} = 0 \) and the other with \( \frac{dx(t)}{dt} = 0 \), then the intersection will be a stable equilibrium point, since both stock and price will no longer have any pressure to move.\(^{26}\)

I first search for equilibriums and nullclines by setting each equation in the system equal to 0 and then solving for price. This technique yields nullclines where either \( \frac{dp(t)}{dt} \) or \( \frac{dx(t)}{dt} \) are zero; barring any shocks to the system, the price and stock will want to stay constant.

The next set of nullclines were derived by solving for price, \( p \), where the denominators of \( \frac{dp(t)}{dt} \) and \( \frac{dx(t)}{dt} \) equal zero. These lines represent values where \( \frac{dp(t)}{dt} = \infty \) or \( \frac{dx(t)}{dt} = \infty \). Mathematically though, the system will never move onto these lines, as they represent nonsensical values where an equation is divided by 0. However, due to the large changes in stock and/or price, the system gets infinitely close to those points without actually attaining them.

Figure 7.1 approximates the relevant features of the resulting phase plane. The lines represent movements in the different quadrants as well as along the lines. We know

\(^{26}\) At all points in the phase plane the Quantity Supplied will equal the Quantity Demand and fishermen will maximize their Utility, since these conditions were assumed in the model's specification.
this to be the relevant area from the data used to derive the demand curve; price was about $1 per pound and the stock must be between 0 and the chosen maximum sustainable yield ($k$).

I found three nullclines in the relevant area of the phase plane. The highest line, line A, is a convenient upper bound for the range of prices worth examining. At line A, $\frac{dp(t)}{dt} = 0$. Given my parameters, the equation for line A is:

\[ \frac{dp(t)}{dt} = 0 \]
\[
p = \frac{2133}{2500} \frac{5107205107}{8 \cdot 10^{19}} \cdot x + \frac{1}{8 \cdot 10^{19}} \cdot \sqrt{(4.6599119370304 \cdot 10^{39} - 6.97194783566784 \cdot 10^{29}) \cdot x + 26083544004966881449 \cdot x^3}
\]

Line B represents points where \( \frac{dp(t)}{dt} = \infty \). Points near the line are pushed towards the stock value \( x = E_v \) according to the phase planes dynamics. The movement of these values does not occur strictly along line B, since values on the curve are not mathematically sensible (values along line B require dividing by 0).

Points near line B and below the stock level \( x = E_v \) experience an increase in stock and either an increase or decrease in price, depending on the price at the initial value. If the initial value of price is above line B, it will increase over time. If it is below it will decrease. For values near line B and above the stock value \( x = E_v \), the price is pushed towards line B so the system will seem to move towards the stock level \( x = E_v \) along line B. Given my parameters, the equation for line B is:

\[
p = -2.631578947 \cdot 10^{-18} \cdot \frac{-5 \cdot 10^{25} - 3.24216 \cdot 10^{19} \cdot x + 1.212961213 \cdot 10^9 \cdot x^2}{x}
\]

Line C is a function derived by setting \( \frac{dx(t)}{dt} = 0 \). It has a vertical asymptote at \( E_v \). For values along line C but below \( E_v \), the pressure on price is negative. For values above \( E_v \) the pressure on price is positive. This means that, even though there is no pressure on stock to change at points on line C, the price will still be forced to change, which will cause the system to continue to adjust. The equation for line C, given my parameters, is:

\[
p = 5 \cdot 10^8 \cdot \frac{-1.274180896 \cdot 10^{11} + 1269 \cdot x}{x \cdot (-2.387511458 \cdot 10^{13} + 24111 \cdot x)}
\]

For simplicity my further analysis of this system focuses on the most relevant values. Given the data used for the derivations of my parameters, these values are the
four quadrants shown in Figure 7.2. It is these quadrants that my following analysis refers to by number.

Points in QI seek equilibrium where lines B and C meet. In QI, if the stock drops, *ceteris paribus*, fishermen will work more hours, total revenues drop, total costs will increase, fishermen will exit the industry, and the quantity supplied by both individual fishermen and the industry will decrease. If stock drops then fishermen need to work more hours to catch the same amount of fish. As they work more, they incur more marginal costs, driving up total costs. Profits drop as a result and fishermen exit the industry. The lower efficiency of fishermen, due to a lower fish density, and the fewer fishermen in the industry, end up being stronger influences on the harvest than the increase in hours worked.

If price drops, *ceteris paribus*, the same changes occur, except in the case of quantity supplied. Quantity supplied increases for both the industry and the individuals when price drops. When price drops, fishermen again react by working more so as to
cover fixed costs. Again marginal costs go up, profits down, and fishermen exit the industry. However, in this case the increased hours worked actually outweighs the negative effects on quantity supplied of fewer fishermen. The difference is that, initially, the stock has not also dropped so the harvesting efficiency of fishermen has remained the same.

The trend in QI is for both the price and stock to drop as the system moves toward equilibrium. Equilibrium is found at the intersection of lines B and C where $\frac{dp(t)}{dt} = 0$ and $\frac{dx(t)}{dt} = 0$. Although the system never moves onto the asymptote, it essentially moves infinitely close to the intersection and, barring shocks to the system, will stay there. The result of these simultaneous movements is that fishermen work more hours, some fishermen exit the industry as revenue drops and costs rise, and the quantity supplied by both fishermen and the industry decreases. Thus, the effects of the decrease in stock on the quantity supplied outweigh the effects of the decrease in price.

In QI, the quantity supplied decreases over time for two reasons. First, as the stock of fish decreases, the productivity from an hour of fishing decreases because of lower fish density. Also, since fishermen are leaving the industry, there are fewer boats to catch fish. This outweighs the effects of fishermen working more, which they chose to do because the price has dropped.

Over time, the price in QI also decreases. As the price drops, effort increases and, even though fishermen exit the industry, the total quantity supplied feels pressure to increases. This further causes the price to drop. However, as the stock decreases, the harvesting efficiency of fishermen decreases, due to lower population density, and there is an overall lower quantity supplied. This provides a positive pressure on price. In QI the negative pressure governs the movement to equilibrium. However, as the system adjusts to equilibrium, the magnitude of the positive pressure on price increases relative to the magnitude of the negative pressure because the stock continues to drop. These pressures balance out when line B is met.

Table 7.1 provides some approximates of relevant points used in this analysis:
Table 7.1

<table>
<thead>
<tr>
<th>p</th>
<th>x (*10^8)</th>
<th>w</th>
<th>n</th>
<th>TR</th>
<th>TC</th>
<th>Q</th>
<th>Q</th>
<th>dx/dt</th>
<th>dp/dt</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.2</td>
<td>84.07</td>
<td>137.43</td>
<td>3833.97</td>
<td>91.078</td>
<td>383436</td>
<td>2790</td>
<td>-241860</td>
<td>-0.00499</td>
</tr>
<tr>
<td>1</td>
<td>1.3</td>
<td>84.072</td>
<td>140.62</td>
<td>4153.17</td>
<td>91.072</td>
<td>415358</td>
<td>2953</td>
<td>-386679</td>
<td>-0.00729</td>
</tr>
<tr>
<td>1.2</td>
<td>1.2</td>
<td>84.065</td>
<td>145.09</td>
<td>4600.05</td>
<td>91.065</td>
<td>383375</td>
<td>2642</td>
<td>-241799</td>
<td>-0.00175</td>
</tr>
</tbody>
</table>

Table 7.2 gives the corresponding approximate values at points along an equilibrium path. The values for this table are arrived from first specifying an initial condition and then adding the \( \frac{dx}{dt} \) and \( \frac{dp}{dt} \) at that point to the point’s p and x values to determine the next point.

Table 7.2

<table>
<thead>
<tr>
<th>P</th>
<th>x (*10^8)</th>
<th>w</th>
<th>n</th>
<th>TR</th>
<th>TC</th>
<th>Q</th>
<th>q</th>
<th>dx/dt</th>
<th>dp/dt</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.2</td>
<td>84.07</td>
<td>137.43</td>
<td>383</td>
<td>91.07</td>
<td>38343</td>
<td>279</td>
<td>-</td>
<td>24186</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>3</td>
<td>3</td>
<td>8</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>-0.0049</td>
<td>0.0049</td>
</tr>
<tr>
<td>.99501</td>
<td>1.1975</td>
<td>84.07</td>
<td>137.1</td>
<td>380</td>
<td>91.07</td>
<td>38266</td>
<td>278</td>
<td>-</td>
<td>23859</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>9</td>
<td>6</td>
<td>7</td>
<td>5</td>
<td>9</td>
<td>9</td>
<td>0.0051</td>
<td>2</td>
</tr>
</tbody>
</table>

All approximations were made for the benefit of the tables and do not interfere with the analysis of variable movements. All four quadrants were analyzed in this manner.

Also, as the stock and price decrease, the rate of change of \( \frac{dx(t)}{dt} \) decreases and the rate of change of \( \frac{dp(t)}{dt} \) increases. Both of these effects reinforce the trend in stock and price illustrated in Figure 7.3. As the system moves toward equilibrium in QI, the change in price begins to dominate the effect, and the change in stock matters less and
less until the system nearly moves vertically to line B. If the price is high enough at the
initial condition, though, the system may move to line C instead. Figure 7.3 illustrates
both types of movements with dashed lines.

Since the rate of growth of the fish stock is dependant on the current stock
population, per the logistic model, as the stock drops in QI, the rate of the stocks growth
will increase. Thus, we may have an economic explanation for the asymptote in line C.
This is where the magnitude of the harvest and the magnitude of the stock’s growth
intersect. At stock levels below the asymptote the fish stock grows faster than it can be
harvested. Once it passes the asymptote, it slows down enough for the harvesting to
force negative growth. This is illustrated in Figure 7.4.
Mathematically, since I set the productivity of fishermen to be close to the maximum sustainable yield (the marginal effects of compounding were ignored), the existence of such an asymptote could be expected. As stock decreases, the rate of stock growth increases and the efficiency of fishermen decreases. Intuitively, at low stock levels, the growth of the fish stock should motivate stock movements more than the harvest rate. As the stock approached its biological carrying capacity, its rate of growth declines and the per hour efficiency of fishermen increases. Intuitively, again, there may be a stock size where the stock stops its tendency to grow and begins to shrink for all larger values. This stock size is the asymptote in line C.

QIV has many conflicts between the influences of a decline in stock and a rise in price, unlike QI, where the only conflict along the equilibrium path was the quantity supplied. In QIV, as stock drops, *ceteris paribus*, there are increases in hours-worked and total costs. There are decreases total revenue, the number of fishermen (due to lower profits), the quantity supplied by industry and individual fishermen, and $\frac{dx(t)}{dt}$ and
\[
\frac{dp(t)}{dt}
\] As price rises, ceteris paribus, there are declines in hours worked, total costs, quantity supplied by the industry and the individual fishermen, and \(\frac{dx(t)}{dt}\). There are increases in the number of fishermen, total revenue, and \(\frac{dp(t)}{dt}\). Between these two tendencies there are conflicts in the movement of hours worked, the number of fishermen, total revenue, total costs, and \(\frac{dp(t)}{dt}\). When the stock drops, ceteris paribus, harvesting efficiency decreases, which drops total revenue. To compensate, some fishermen leave the industry and others begin to work more, (which increases costs). The net effect, though, is a drop in the quantity supplied. When price increases, ceteris paribus, fishermen work less (which drives costs down) and yet still experience an increase in revenue. Also, people enter the industry. However, the effect of each fisherman working less dominates and the quantity supplied from the individuals and the industry both drop.

In QIV, as we move to equilibrium, hours worked and total costs increase while the number of fishermen, total revenues, quantity supplied by the industry, quantity supplied by individual fishermen, \(\frac{dx(t)}{dt}\), and \(\frac{dp(t)}{dt}\) all decreased. In all cases of conflicting tendencies, the effects of a decreasing stock were greater than the effects of the rise in price.

The first result is that the stock decreases until the decreased efficiency of fishermen, caused by lower population density, is offset by the rise in the rate of stock growth, as specified per the logistic model. As discussed above, this corresponds to the asymptote of line C. The second result is that the price rises until the positive effect on price from the decrease in quantity supplied is offset by the negative effect on price from a lower stock, which causes fishermen to work more until they succeed in catching enough fish to halt the decreasing quantity supplied.

Figure 7.5 illustrates the possible movements to equilibrium in QIV.
QII and QIII are symmetries of the analysis that occurred in QIV and QI. Thus, in QII, stock will increase to the asymptote where the effect of a low stock on the growth rate is balanced by the effect of a denser stock on the harvesting efficiency. Also, price will rise until the decrease in hours worked, and the corresponding decrease in quantity supplied, is counterbalanced by the entry of new fishermen into the industry. In QIII, again we see stock being forced towards the vertical asymptote and price being forced down for reasons that mirror QI.
Dynamic Shifts

Having constructed the model and explained the logic governing the movements of stock and price, I can now examine what happens when the model experiences dynamic shifts. These are shocks to the parameters that fundamentally change the phase plane and cause all conditions to follow a new path to a new equilibrium, even if they initially were at equilibrium. These shocks can be demonstrated by changing the values I used for my parameters. All graphs assume that equilibrium is the initial condition.

<table>
<thead>
<tr>
<th>Key</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original Phase Lines:</td>
</tr>
<tr>
<td>New Phase Lines:</td>
</tr>
<tr>
<td>New Paths to Equilibrium:</td>
</tr>
</tbody>
</table>

Changes in the Intrinsic Growth Rate ($r$)

As $r$ increases the vertical asymptote of line C moves right until it approaches the maximum sustainable population, which it does not cross. The result is that the stock of fish will have a greater tendency towards equilibrium. With a high enough intrinsic growth rate the system will tend towards the MSY.\(^{27}\)

Figure 8.1 demonstrates this movement. The movements in lines $A$ and $B$ are negligible have been ignored for simplicity in this graph.

---

\(^{27}\) Since, in reality, a system is constantly being shocked out of any equilibrium, the true equilibrium for stock has always been the asymptote. That is, if the system is at equilibrium anywhere on line B, almost any shock will cause it to move closer to the asymptote. Exceptions include unlikely shocks such as a sudden increase in the stock of fish.
However, more likely than a sudden increase in the intrinsic growth rate of fish is a decrease, from such causes as pollution or climate changes. In this case the asymptote would move left, and the system would seek an equilibrium at a lower value of $x$. Should the rate drop too far the stock will go to zero over time and the fishery will be destroyed. Graphically, this is simply the reverse of the dynamic response to an increase in $r$.

Changes in the Efficiency of Fishermen ($g$)

A decrease in $g$ works similar to an increase in $r$ in that the asymptote is pushed to the MSY. However, the system also collapses inward and leftward so that new quadrants become of interest should the change be on a magnitude of ten times greater efficiency.

Figure 8.2 illustrates the resulting graph for such a drastic decrease in $g$ and some corresponding shifts to equilibrium.
An increase in $g$ shifts the vertical asymptote left until it becomes irrelevant and the system always moves to equilibrium through decreases in stock. Thus, eventually such a system would drive stock to zero, again destroying the fishery. Figure 8.3 illustrates this system.
It turns out that the ratio between $r$ and $g$ is the main determinate of equilibrium. If one is too high, relative to the other, its effect will dominate and either drive the stock to the MSY or to zero. When the ratio is managed so that fishermen could, at most, only remove about the MSY in a year, the phase diagram has equilibrium points at the intersection of lines $B$ and $C$ as I found.

Changes in Demand ($a$)

Increases in $a$ increase the $p$ values of lines $A$ and $B$ so that, for the relevant values, line $C$ becomes the only relevant line. In other words, as $a$ increases, QIII and QIV expand until they are the only relevant quadrants. Adjustments to equilibrium thus follow the explanations given previously for those quadrants.

Decreases in $a$ have the opposite effect, lowering lines $A$ and $B$ until the only relevant quadrants are those two above line $A$. However, adjustments to this new phase plane will be the same as the previous plane in that the system tends towards the vertical asymptote.
Changes in Marginal Costs ($f$)

Increases in marginal costs mimic the results of a decrease in $g$, which is diagrammed in Figure 8.2, except that the vertical asymptote does not move in this case.

Decreases in marginal costs only seem to flatten the curves in lines A, B, and C so that, with the exception of the asymptote which has not moved, they look nearly horizontal. Dynamics simply follow the logic of whatever quadrant the equilibriums ended up in. Essentially though, this just further motivates the system to find an equilibrium for stock at the vertical asymptote. Marginal costs do not greatly seem to affect the phase plane.

Changes in Fixed Costs ($f$)

Changes in fixed costs may cause line C to fundamentally alter the phase plane. Notably, QIII shrinks and the tendencies of QIV push into its area, as shown in Figure 8.4. Furthermore, line D, which previously has always traced line B, has come into its own and now is relevant. Figure 8.4 shows the change assuming a 100-times increase in fixed costs.
When fixed costs are lowered, the phase plane does not change to any significant degree. Line C simply drops lower and the original quadrants are intact.

Changes in the Elasticity of Utility with Respect to Income ($c_y$) and Leisure ($c_l$)

These variables mimic each other: a decrease in one corresponds to an increase in the other. If $c_y$ is decreases or if $c_l$ is increased the vertical asymptote moves right until it hits the MSY. This mimics Figure 8.1. If the opposite occurs, then the asymptote moves left, causing the equilibrium stock to be lower.

These variables affect the preferences of fishermen for work or leisure. As the elasticity of utility with respect to income drops, fishermen will choose to work less because, given their previous equilibrium, the marginal utility of leisure relative to income will have increased. Likewise, if the elasticity of utility with respect to leisure should increase, fishermen will work with the same motivations. Opposite changes in these variables have opposite effects.
IX. Conclusion

In conclusion I have several observations on the model, its implications, and the possibilities for future research. First, it seems that if the efficiency of fishermen does not overwhelm the intrinsic growth rate of the fish, then there will be an equilibrium that the system moves towards. This has been represented as the intersection of lines B and C. If the efficiency is too great, however, the stock of fish will decrease to zero.

Boycotts, fish farming, and anything else that decreases the consumers demand for fish caught in an open-access fishery affect the position of line B. They do not change the equilibrium stock level, which is maintained at $E_v$. If the system is already at equilibrium, a decrease in demand, $a$, will cause the price to move down along the asymptote of line C to the new equilibrium; the stock of fish changes very little while the price drops.

If the system is in QI and not at equilibrium when demand drops, then the stock of fish will begin to drop faster on its approach to the equilibrium stock because fishermen will supply a greater quantity of fish. The ultimate equilibrium stock level, though, does not change significantly. Thus, lower demand does not significantly affect the long run equilibrium stock level.

If, on the other hand, no stable equilibrium exists (perhaps harvesting efficiency is too great for the system), then the stock is moving to zero over time anyways. However, a decrease in demand, by encouraging a larger quantity of fish to be supplied, will accelerate the decrease in stock and give policymakers less time to save the fishery.

Thus, to significantly increase the long-run stock level of a fishery, the vertical asymptote of line C must be increased. Through regulation this can be done with a decreased fishing season. In this model, setting the variable $u$ in the quantity-supplied equation such that $0 < u > 1$ will act as a restriction on the length of the fishing season by decreasing the amount of fish that can possibly be caught in a year. Since this value is essentially alters $g$, changes in $u$ have the same dynamics as changes in $g$, which change the long-run stock equilibrium.
The model does have several weak aspects that future research may want to remedy. First, the estimation of the demand curve is poor, as noted before. It takes real world data and assumes that they reflect a single demand curve, whereas the data may in fact reflect several different curves.

The model also sets the investment decision of fishermen exogenously. In fact, investment should be a function of current profits, the interest rate, and the expected future income. Making it endogenous would more realistically model the decision making process as investment should vary with the current price and stock. This could also help model the overcapitalization of fisheries that occurs when regulations are placed on the length of the fishing season. My model does not show the economic inefficiencies of overcapitalization. In any new model it would also be helpful to let the harvesting efficiency of fishermen vary with investment, so that new equipment can increase efficiency.

Also, the decision of fishermen to enter or exit is set to be a function of profits. As profits increase fishermen enter and as profits decrease they exit. My specification of this behavior gives fishermen the same propensity to enter and exit the industry. Instead, it may be more realistic to make fishermen more reluctant to leave the industry, given the barriers to exit, than to enter the industry.

I also have not examined the model for the time it takes to adjust to equilibrium. At times, it seemed like the system slowed as it approached equilibrium, making me wonder how the time dynamics change throughout the phase plane. Future research could also use real world parameters to test the model's predictive validity. This would be the most useful evaluation of the model.

To summarize, the general conclusion is largely positive. Demand policies cannot lower the equilibrium level of stock, although they will encourage faster destruction of a fishery when the equilibrium stock is zero.
Maple Appendix

The two most relevant Maple documents were included. This first document was used to derive the system of equations:

```maple
> # Andre Garron
> # Model 6 derivation
> # equations where:
> # g only refers to a part of x
> # mts is changed to lbs
> # time is changed to 1 week
> # C-D utility function
> # n is linearly endogenous
> # 4/28/04
> restart;
> with(DEtools):with(linalg):with(plots):
> # Units of measure
> # F(x): growth of fish stock in lbs
> # h(x): harvest of fish in lbs
> # r: rate of change of fish stock due to births and deaths per week
> # x: lbs of fish in ocean
> # k: max stock of fish in lbs
> # p: price of fish
> # a: in demand equation
> # b: in demand equation
> # Q: total lbs of fish harvested by the industry
> # n: number of fishermen
> # q: lbs of fish harvested per boat
> # U: utility
> # Y: income
> # L: hours of leisure per week
> # TR: Total revenue per week
> # TC: Total costs per week
> # w: hours worked per week
> # f: marginal costs per hour
> # II: fixed costs per week
> # g: percent of available stock caught in one week
> # u: percent of x on which g relies
> # cd: efficiency parameter in C-D utility function
> # cl: elasticity of utility with respect to leisure
> # cy: elasticity of utility with respect to income
> # j1: number of ships at economic profits
> # j2: propensity of ships to enter or leave industry
> eq1:=diff(x(t),t)=F(x(t))-h(x(t));
```
\[ F_1 := r \cdot x(t) \cdot (1 - (x(t)/k)) \]
\[ eq2 := p(t) = a - b \cdot Q(t) \]
\[ Q_1 := n \cdot q \]
\[ VI := c \cdot d \cdot Y \cdot (c) \cdot (d) \]
\[ Y_I := TR - TC \]
\[ TRI := q \cdot p(t) \]
\[ TC_I := w \cdot f + II \]
\[ q_1 := g \cdot x(t) \cdot w \cdot u \]
\[ TR_2 := \text{subs}(q = q_1, TR_1) \]
\[ Y_2 := \text{subs}(TR = TR_2, TC = TC_1, Y_1) \]
\[ n_2 := \text{subs}(Y = Y_2, n_1) \]
\[ Q_2 := \text{subs}(n = n_2, Q_1) \]
\[ L_1 := 168 - w \]
\[ U_2 := \text{subs}(Y = Y_2, L = L_1, U_1) \]
\[ dU := \text{diff}(U_2, w) \]
\[ w_1 := \text{solve}(dU = 0, w) \]
\[ q_2 := \text{subs}(w = w_1, q_1) \]
\[ Q_3 := \text{subs}(q = q_2, Q_2) \]
\[ Q_4 := \text{subs}(w = w_1, Q_3) \]
\[ eq3 := \text{diff}(x(t), t) = F_1 - Q_4 \]
\[ eq4 := \text{subs}(Q(t) = Q_4, eq2) \]
\[ eq6 := \text{simplify}(eq4) \]
\[ eq7 := (\text{lhs}(eq4) \cdot \text{denom}(\text{rhs}(eq4)) = \text{numer}(\text{rhs}(eq4))) \]
\[ eq8 := \text{diff}(eq7, t) \]
\[ eq9 := \text{diff}(p(t), t) = \text{solve}(eq8, \text{diff}(p(t), t)) \]
\[ # \text{System of Equations} \]
\[ eq3 ; \]
\[ eq9 ; \]
\[ delp := \text{subs}(x(t) = x, p(t) = p, eq9) ; \]
\[ delx := \text{subs}(x(t) = x, p(t) = p, eq3) ; \]
\[ # \text{formula for } w \text{ given an } x \text{ and } p : \]
\[ w_1 ; \]
\[ # \text{formula for } n \text{ at these points is :} \]
\[ eq14 := n = j_1 + j_2 (g \cdot x(t) \cdot w \cdot u \cdot p(t) - w \cdot f - II) ; \]
\[ # \text{total revenue formula} \]
\[ eq15 := tr = g \cdot x(t) \cdot w \cdot u \cdot p(t) ; \]
\[ # \text{total cost formula} \]
\[ eq16 := tc = w \cdot f + II ; \]
\[ # \text{equation for } Q \]
\[ Q_2 := (j_1 + j_2 (g \cdot x(t) \cdot w \cdot u \cdot p(t) - w \cdot f - II)) \cdot (g \cdot x(t) \cdot w \cdot u) ; \]
\[ Q_3 := Q = \text{subs}(w = w_1, Q_2) ; \]
This second document is used to determine the nullclines, equilibrium, analyze dynamic shocks to the system, and graph the phase plane:

> # Andre Garron
> # Model 6 Analysis
> # g is a part of x
> # mts is changed to lbs
> # time is in weeks
> # Cobb Douglas utility function
> # n is endogenous (linear)
> # 4/28/04
> restart;
> with(plots):with(DEtools):

\[ \text{delp} := -g \cdot u \cdot \text{delx} \cdot (-168 \cdot b \cdot j1 \cdot cy^2 \cdot f + cl \cdot b \cdot j1 \cdot II \cdot cy + cl \cdot 2 \cdot b \cdot j1 \cdot II - 168 \cdot cl \cdot b \cdot j1 \cdot cy \cdot f - cl \cdot 2 \cdot b \cdot j2 \cdot (168 \cdot g \cdot x \cdot u \cdot p - 168 \cdot f - II) \cdot cy/(cy + cl)) - II - 168 \cdot cl \cdot b \cdot j2 \cdot (168 \cdot g \cdot x \cdot u \cdot p - 168 \cdot f - II) \cdot cy/(cy + cl) \cdot cy \cdot f - 168 \cdot b \cdot j2 \cdot (168 \cdot g \cdot x \cdot u \cdot p - 168 \cdot f - II) \cdot cy/(cy + cl) \cdot cy \cdot f + cl \cdot b \cdot j2 \cdot (168 \cdot g \cdot x \cdot u \cdot p - 168 \cdot f - II) \cdot cy/(cy + cl) \cdot cy \cdot f;
\]

\[ \text{delx} := r \cdot x \cdot (1-x/k)-(j1 + j2(g \cdot x \cdot (168 \cdot cy \cdot g \cdot x \cdot u \cdot p - 168 \cdot cy \cdot f + cl \cdot II)) \cdot u \cdot p/(cy \cdot g \cdot x \cdot u \cdot p - cy \cdot f + cl \cdot II \cdot f)(168 \cdot cy \cdot g \cdot x \cdot u \cdot p - 168 \cdot cy \cdot f + cl \cdot II) \cdot u/cy \cdot g \cdot x \cdot u \cdot p - cy \cdot f + cl \cdot II \cdot f) \];

> delps := factor(simplify(delps));
> delxs := factor(simplify(delxs));
> delps := factor(simplify(delx));

> # Parameters
> # nice parameters
> # excellent- no significant change

50
> # j2 not equal to 0
> # fO:=1; b0:=0.00000002; II0:=7; g0:=0.00006410; a0:=1.7064; r0:=.33; k0:=132300000; u0:=1;
c0:=1; c10:=1; cy0:=1; j10:=100; j20:=0.0001; xmin:=0; pmin:=0; xrange:=200000000; prange:=2;
> # newly derived parameters
> # fO:=1; b0:=0.00000002; II0:=7; g0:=0.00006410; a0:=1.7064; r0:=.01269; k0:=132300000; u0:=
> =1; c0:=1; c10:=1; cy0:=1; j10:=100; j20:=0.0001; xmin:=0; pmin:=0; xrange:=200000000; prange:=2;
> # basic new diagram - which is nice b/c x can increase
> > # new parameters with a short range
> # fO:=1; b0:=0.00000002; II0:=7; g0:=0.00000038; a0:=1.7064; r0:=.01269; k0:=132300000; u0:=
> =1; c0:=1; c10:=1; cy0:=1; j10:=100; j20:=0.0001; xmin:=0; pmin:=0; xrange:=1.01e+08; prange:=1;
> # looks consistent, but note the orange line to the left of the vertical equilibria
> # increase f
> # fO:=1; b0:=0.00000002; II0:=7; g0:=0.00000038; a0:=1.7064; r0:=.01269; k0:=132300000; u0:=
> =1; c0:=1; c10:=1; cy0:=1; j10:=100; j20:=0.0001; xmin:=0; pmin:=0; xrange:=200000000; prange:=2;
> # the blue curve moves up and becomes relevant
> > # decrease f
> # fO:=1; b0:=0.00000002; II0:=7; g0:=0.00000038; a0:=1.7064; r0:=.01269; k0:=132300000; u0:=
> =1; c0:=1; c10:=1; cy0:=1; j10:=100; j20:=0.0001; xmin:=0; pmin:=0; xrange:=200000000; prange:=2;
> # the blue curve becomes very irrelevant and the curves flatten
> > # increase b
> # fO:=1; b0:=0.00000002; II0:=7; g0:=0.00000038; a0:=1.7064; r0:=.01269; k0:=132300000; u0:=
> =1; c0:=1; c10:=1; cy0:=1; j10:=100; j20:=0.0001; xmin:=0; pmin:=0; xrange:=200000000; prange:=2;
> # nothing much
> > # decrease b
> # fO:=1; b0:=0.00000002; II0:=7; g0:=0.00000038; a0:=1.7064; r0:=.01269; k0:=132300000; u0:=
> =1; c0:=1; c10:=1; cy0:=1; j10:=100; j20:=0.0001; xmin:=0; pmin:=0; xrange:=200000000; prange:=2;
> # nothing much
> > # increase II
> # fO:=1; b0:=0.00000002; II0:=70; g0:=0.00000038; a0:=1.7064; r0:=.01269; k0:=132300000; u0:=
> =1; c0:=1; c10:=1; cy0:=1; j10:=100; j20:=0.0001; xmin:=0; pmin:=0; xrange:=200000000; prange:=2;
> # cyan line goes away?
> > # decrease II
> #f0:=1;b0:=.00000002;\Pi 0:=7;g0:=.00000038;a0:=1.7064;r0:=.01269;k0:=132300000;u
0:=1;cd0:=1;c0:=1;cy0:=1;j10:=100;j20:=.0001;xmin:=0;pmin:=0;xrange:=200000000;prange:=2;
> # no real change
>
> # increase g
> #f0:=1;b0:=.00000002;\Pi 0:=7;g0:=.00000038;a0:=1.7064;r0:=.01269;k0:=132300000;u0:=
0:=1;cd0:=1;c0:=1;cy0:=1;j10:=100;j20:=.0001;xmin:=0;pmin:=0;xrange:=200000000;prange:=2;
> # becomes the precursor to our former swirl (the no swirl) but then becomes the swirl if
increased by a factor of 100
>
> # decrease g
> #f0:=1;b0:=.00000002;\Pi 0:=7;g0:=.00000038;a0:=1.7064;r0:=.01269;k0:=132300000;u
0:=1;cd0:=1;c0:=1;cy0:=1;j10:=100;j20:=.0001;xmin:=0;pmin:=0;xrange:=200000000;prange:=2;
> # similar to if i raised f
>
> # increase a
> #f0:=1;b0:=.00000002;\Pi 0:=7;g0:=.00000038;a0:=1.7064;r0:=.01269;k0:=132300000;u
0:=1;cd0:=1;c0:=1;cy0:=1;j10:=100;j20:=.0001;xmin:=0;pmin:=0;xrange:=200000000;prange:=2;
> # only blue line seems to become relevant
>
> # decrease a
> #f0:=1;b0:=.00000002;\Pi 0:=7;g0:=.00000038;a0:=1.7064;r0:=.01269;k0:=132300000;u
0:=1;cd0:=1;c0:=1;cy0:=1;j10:=100;j20:=.0001;xmin:=0;pmin:=0;xrange:=200000000;prange:=2;
> # the other curves dip so low that again only blue is relevant
>
> # increase r
> #f0:=1;b0:=.00000002;\Pi 0:=7;g0:=.00000038;a0:=1.7064;r0:=.01269;k0:=132300000;u
0:=1;cd0:=1;c0:=1;cy0:=1;j10:=100;j20:=.0001;xmin:=0;pmin:=0;xrange:=200000000;prange:=2;
> # verticle equilibrium moves right
>
> # decrease r
> #f0:=1;b0:=.00000002;\Pi 0:=7;g0:=.00000038;a0:=1.7064;r0:=.01269;k0:=132300000;u
0:=1;cd0:=1;c0:=1;cy0:=1;j10:=100;j20:=.0001;xmin:=0;pmin:=0;xrange:=200000000;prange:=2;
> # no visible verticle equilibrium. prob moved left too far.
>
> # increase cd
> #f0:=1;b0:=.00000002;\Pi 0:=7;g0:=.00000038;a0:=1.7064;r0:=.01269;k0:=132300000;u
0:=1;cd0:=10;c0:=1;cy0:=1;j10:=100;j20:=.0001;xmin:=0;pmin:=0;xrange:=200000000;prange:=2;

52
> # no real effect
>
> # decrease cd
>
> # £0 := 1; b0 := 0.00000002; a0 := 1.7064; r0 := 0.01269; k0 := 132300000; u0 := 1; cd0 := 1; cl0 := 1; cy0 := 1; j10 := 100; j20 := 0.0001; xmin := 0; pmin := 0; xrange := 200000000; prange := 2;
> # no real effect
>
> # increase cl
>
> # decrease cl
>
> # increase cy
>
> # decrease cy
>
> # cy, cl = .5
>
> # looks like original
>
> # cy = .25, cy = .25
>
> # looks about like original
>
> # increase j2

53
> # fO:=1, b0:= .00000002, II0:= 7, g0:= .00000038, a0:= 1.7064, r0:= .01269, k0:= 132300000, u0:= 1, cd0:= 1, cl0:= 1, cy0:= 1, j10:= 100, j20:= .001, xmin:= 0, pmin:= 0, xrange:= 200000000, prange:= 2;
> # nothin really
> # decrease j2
> # fO:=1, b0:= .00000002, II0:= 7, g0:= .00000038, a0:= 1.7064, r0:= .01269, k0:= 132300000, u0:= 1, cd0:= 1, cl0:= 1, cy0:= 1, j10:= 100, j20:= .00001, xmin:= 0, pmin:= 0, xrange:= 200000000, prange:= 2;
> # nothin really
> # substitute parameters
> delx0:= subs(f=fO, b=b0, II=II0, g=g0, a=a0, r=r0, k=k0, u=u0, cd=cd0, cl=cl0, cy=cy0, j1= j10, j2=j20, delxs);
> delp0:= subs(f=fO, b=b0, II=II0, g=g0, a=a0, r=r0, k=k0, u=u0, cd=cd0, cl=cl0, cy=cy0, j1= j10, j2=j20, delp0);
> # solve for p
> delx1 := solve(delx0=0, p);
> delp1 := solve(delp0=0, p);
> infx := solve(denom(delx0)=0, p);
> infp := solve(denom(delp0)=0, p);
> phase0 := fieldplot([diff(x(t), t)=delx0, diff(p(t), t)=delp0], [x(t), p(t)], t=0..5, x=xmin..xrange, p=pmin..prange, arrows=slim);
> delx1p := plot(delx1p, x=xmin..xrange, y=pmin..prange, color=blue);
> delp1p := plot(delp1p, x=xmin..xrange, y=pmin..prange, color=magenta);
> delp1bp := plot(delx1bp, x=xmin..xrange, y=pmin..prange, color=green);
> # note: delp1cp is not plotted because it equals delx1p
> infxp := plot(infxp, x=xmin..xrange, y=pmin..prange, color=orange);
> # note: infpap is not plotted because it equals infxp
> infpbb := plot(infp[2], x=xmin..xrange, y=pmin..prange, color=cyan);
> # idea: also include x values that make sqrt impossible for delp0
> display(phase0, delx1p, delp1p, infxp, infpbb);
> # Line A1 analysis
> x0 := 6e+07;
> p0 := subs(x=x0, delp1[1]);
> PlineA1 := subs(x=x0, p=p0, delp0);
> XlineA1 := subs(x=x0, p=p0, delx0);
> # Line A2 analysis
> x0 := 1.2e+08;
> p0 := subs(x=x0, delp1[1]);
> PlineA2 := subs(x=x0, p=p0, delp0);
> XlineA2 := subs(x=x0, p=p0, delx0);
> # Line B1 analysis
> x0 := 6e+07;
> p0 := subs(x=x0, p=p0, infp[2]);
> PlineB1 := subs(x=x0, p=p0, delp0);
Error, division by zero
> XlineB1:=subs(x=x0,p=p0,delx0);
> # Line B2 analysis
> x0:=1.2e+08:
> p0:=subs(x=x0,inf[p[2]]);
> PlineB2:=subs(x=x0,p=p0,delp0);
Error, division by zero
> XlineB2:=subs(x=x0,p=p0,delx0);
> # Line C1 Analysis
> x0:=6e+07:
> p0:=subs(x=x0,delx1);
> PlineC1:=subs(x=x0,p=p0,delp0);
> XlineC1:=subs(x=x0,p=p0,delx0);
> # Line C2 Analysis
> x0:=1.2e+08:
> p0:=subs(x=x0,delx1);
> PlineC2:=subs(x=x0,p=p0,delp0);
> XlineC2:=subs(x=x0,p=p0,delx0);
Resources Used


