On property Pm,n and some applications to graph theory

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Colby College
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ABSTRACT

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An intriguing concept in convexity theory is the characterization of sets having property $P_{m,n}$. A set is said to satisfy property $P_{m,n}$ if for any collection of $m$ arbitrary points at least $n$ of the $\binom{m}{2}$ line segments determined by the points is contained in the given set. Valentine did the initial work with property $P_{3,1}$, showing that a set with this property could be expressed as the union of 3 or fewer convex sets. This work was extended by Kay and Guay, who worked with the more general property $P_{m,n}$. Convex sets and graph theory were linked by Guay, and I have sought to continue this work by combining property $P_{m,n}$ and graph theory. A graph $G$ is said to have property $P_{m,n}$ if given any $m$ arbitrary points in $G$, there are at least $n$ edges joining these points, $n < m$.

The aim of this paper is to present some results which were obtained while seeking to characterize graphs having property $P_{m,n}$. The maximum diameter of a graph having property $P_{m,n}$, and the minimum number of edges necessary for property $P_{3,1}$ have been found. In any graph $G$, property $P_{m,n}$ can be extended to property $P_{m+1,n+1}$. Finally, a graph can be generated having no star-complete points. There seem to be several other methods for characterizing graphs having property $P_{m,n}$, but I did not have time to study them.
An intriguing concept in convexity theory is that of the characterization of sets having property $P_{m,n}$. A set is said to satisfy property $P_{m,n}$ if for any collection of $m$ arbitrary points at least $n$ of the $\binom{m}{2}$ line segments determined by the points is contained in the given set. Property $P_m$ is the special case where $n = 1$. The initial work, done by Valentine [8], has shown that a closed connected set in $E^2$ having property $P_3$ can be expressed as the union of 3 or fewer convex sets. The more general property $P_{m,n}$ has been studied by Kay and Guay [6]. They obtained several results concerning conditions necessary for a set to have property $P_{m,n}$. Guay [3] also did some work equating concepts in convex sets with similar concepts in graph theory, and succeeded in producing several theorems which are analogous to some of the major theorems in convexity theory. The combinatorial aspects of property $P_{m,n}$ suggest its application to graph theory. A graph $G$ is said to have property $P_{m,n}$ if given any $m$ arbitrary points in $G$, there are at least $n$ edges joining these points, $n < m$.

The aim of this paper is to present some results obtained from working with graphs having property $P_{m,n}$. So far, no general characterization of graphs having property $P_{m,n}$ has been found, but several interesting results were obtained while
seeking to characterize such graphs. The maximum diameter of a graph having property $P_{m,n}$, and the minimum number of edges necessary for property $P_{3,1}$ have been found. In any graph $G$, property $P_{m,n}$ can be extended to property $P_{m+1,n+1}$. Finally, a graph can be generated having no star-complete points.
Property $P_{m,n}$ and Graph Theory

or

What It Is All About

**Definition:** A graph $G$ consists of a finite nonempty set $V$ of $g$ points together with a prescribed set $E$ of $q$ unordered pairs of distinct points of $V$. Each pair $e = (v_1, v_2)$ of points in $E$ is an edge of $G$, and $e$ is said to join $v_1$ and $v_2$. Further, $e$ is said to be incident with each of the points $v_1$ and $v_2$ and vice versa.

**Definition:** A graph $G$ is said to have property $P_{m,n}$ if given any $m$ arbitrary points in $G$, there are at least $n$ edges joining these points, $n < m$.

**Definition:** A chain $C$ in $G$ is a finite sequence of distinct edges proceeding from one point in $G$ to another point in $G$.

**Definition:** A graph $G$ is connected if every pair of points in $G$ can be linked by at least one chain. All graphs in this paper will be considered connected, unless otherwise stated.

**Definition:** The distance $d(v_1, v_2)$ between any two points, $v_1$ and $v_2$, in $G$, is equal to the number of edges in the shortest chain joining $v_1$ and $v_2$. The diameter $D$ of $G$ is the maximum distance between any two points in $G$.

**Definition:** A subgraph $G_1$ of $G$ is a graph having all its points and edges in $G$. $G_1: (v_1, v_2, v_3, \ldots, v_n)$ will mean the subgraph consisting of the points $(v_1, v_2, v_3, \ldots, v_n)$ with any edges that may exist in $G$ joining them. $G_1$ will be said to be the subgraph generated by $(v_1, v_2, v_3, \ldots, v_n)$. 

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Theorem 1: Given a graph $G$ with $g$ points. If the graph has property $P_{m,n}$, then the diameter of the graph $D \leq 2m-n-2$.

Proof: Assume that there exist two points $v_a$ and $v_c$ belonging to $G$, such that $d(v_a, v_c) = 2m-n-1$. A chain of length $2m-n-1$ has $2m-n$ points, including $v_a$ and $v_c$.

Let $v_a = v_1$ and number the points on the chain to $v_c$ in order, so that $v_c = v_{2m-n}$. The first $m-n$ odd subscripted points, starting with $v_1$, form set $A$, in which no point is joined to any other point. The first $m-n$ even subscripted points form set $B$. The remaining $n$ points, including $v_{2m-n}$ form set $C$. There are $n-1$ edges in the subgraph generated by the points of set $C$. Consider the set of $m$ points formed from sets $A$ and $C$. There are only $n-1$ edges joining the points of the subgraph generated by the points of set $A$ and $C$. By property $P_{m,n}$ however, there must be $n$ edges between these $m$ points. Therefore, at least two of the points of set $A$ must be joined, which will reduce the distance by at least one. Thus $d(v_a, v_c) < 2m-n-1$, and so if $v_a$ and $v_b$ belong to $G$, then $d(v_a, v_b) \leq 2m-n-2$.

Set $A = (v_a, v_3)$
Set $B = (v_2, v_4)$
Set $C = (v_5, v_6, v_c)$

Figure 1

Definition: A graph $G$ with $g$ points is complete if every two distinct points are joined by an edge. The total number of edges in $G$ is $\left(\frac{g}{2}\right)$. 

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**Corollary 1:** The diameter of a graph $G$ with property $P_{m,1}$ is $\leq 2m-3$. That this number is the best possible (i.e. smallest possible) for a graph $G$ with property $P_{m,1}$ is shown by the example having $m-1$ complete subgraphs each joined to the next by a single edge. This graph has property $P_{m,1}$ because there exists no set of $m$ independent points. To show this, start by taking one point from each of the $m-1$ complete subgraphs. This will form a set of $m-1$ independent points. However, since each of these points belongs to a complete subgraph, the last point must see at least one of the other $m-1$ points. This graph has diameter $2m-3$, because each complete subgraph has diameter 1 and each edge joining the subgraphs has diameter 1. Thus $D = 1(m-1) + 1(m-2) = 2m-3$.

**Theorem 2:** Let $G$ be a graph with $g$ points, where $g > m$. If $G$ has property $P_{m,n}$, then it has property $P_{m+1,n+1}$.

**Proof:** Let $A$ be any set of $m+1$ points in $G$. There are $\binom{m+1}{m}$ subsets of $A$ that contain $m$ points. These $m+1$ subsets generate $m+1$ subgraphs, each of which satisfy property $P_{m,n}$. Choose one subgraph, call it $V_0$; $V_0$ has $n$ edges joining its $m$ points. Let $\{x_0\} = A - V_0$. The other $m$ subgraphs of $A$ can now be obtained by replacing, in turn, each of the $m$ points of $V_0$ by $x_0$. In at least one of these subgraphs, call it $V_k$, $x_0$ must replace a point which is incident with at least one of the $n$ edges in $V_0$. For property $P_{m,n}$ to hold, $x_0$ must join at least one of the other $m-1$ points in $V_k$, or the $m-1$ points were already joined by
n edges. Thus the subgraph generated by $A$ has $n+1$ edges.

**Note:** This procedure for extending $P_{m,n}$ can be continued until all $g$ points of $G$ are used, yielding property $P_{g,n+1}(g-m)$. It seems plausible that this property can be extended to $P_{m+1,n}$ where $r > n+1$ but, to date, I have not been able to find a precise formulation for $r$.

**Definition:** The degree of a point $v$ in graph $G$, denoted by $\text{deg}(v)$, is the number of edges incident with $v$.

**Definition:** A dumbbell graph is a graph formed from two complete subgraphs joined by an edge. This graph is a special case of the graph mentioned in Corollary 1, and has property $P_{3,1}$.

**Theorem 3:** The dumbbell graph of $g$ points, in which the number of points in each subgraph differs by at most one, has the least number of edges of all dumbbell graphs with $g$ points.

**Proof:** Consider a dumbbell graph $G$, with $r$ points in one subgraph, and $g-r$ points in the other subgraph. The total number of edges in the graph is

$$E = \binom{r}{2} + \binom{g-r}{2} + 1.$$ 

Evaluating the combinations yields:

$$E = \frac{r(r-1)}{2} + \frac{(g-r)(g-r-1)}{2} + 1$$

$$E = \frac{1}{2}(2r^2 + g^2 - 2gr - g + 2)$$
To minimize the number of edges, take the first derivative with respect to \( r \), (\( g \) is a constant), and set it equal to zero.

\[
\frac{dE}{dr} = \frac{1}{2}(4r - 2g)
\]

\[0 = \frac{1}{2}(4r - 2g)\]

Thus \( 2r = g \) or \( r = \frac{1}{2}g \).

That this value of \( r \) is a minimum can be shown through the second derivative.

\[
\frac{d^2E}{dr^2} = 2 > 0
\]

Therefore, \( r = \frac{1}{2}g \) yields a relative minimum. The actual number of edges in \( G \) is

\[
\begin{cases} 
  n^2 + 1 & \text{if } g = 2n + 1 \\
  n^2 - n + 1 & \text{if } g = 2n
\end{cases}
\]

**Note:** The adjacency matrix \( A = [a_{ij}] \) of a graph \( G \) with \( g \) points is the square matrix of order \( g \), in which \( a_{ij} = 1 \) if \( a_i \) and \( a_j \) are joined by an edge in \( G \), and \( a_{ij} = 0 \) otherwise. The matrix representation of a complete graph \( G \) will have zeros on the diagonal and ones everywhere else.

**Remark:** The graph \( G \) with \( g \) points, described in Theorem 3, can be represented in a matrix \( A \) of order \( g \). \( A \) is formed from two submatrices, each of which represents one of the complete subgraphs. The diagonals of the two submatrices form the main diagonal of \( A \). The other elements in \( A \) will be zeros except for two ones representing the edge joining the
two complete subgraphs.

\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
\]

Figure 2

**Definition:** A cutpoint \( v_p \) of a graph \( G \) is a point whose removal makes the graph disconnected.

**Corollary 3:** The graph \( G \) from Theorem 3, having property \( P_{3,1} \) has the maximum number of cutpoints (2) of any graph with property \( P_{3,1} \).

**Proof:** Assume the graph \( G \) has 3 cutpoints. They can partition \( G \) in one of two ways:

1. They can divide \( G \) into three independent subgraphs, but then property \( P_{3,1} \) would not hold.
2. They could be in a line between the two subgraphs \( G_1 \) and \( G_2 \) which are formed by the cutpoints. Again, there would be an independent set consisting of the middle cutpoint together with any point from \( G_1 \) and \( G_2 \) other than the two
cutpoints, and property $P_{3,1}$ would not hold. Therefore, there can be only two cutpoints in $G$.

**Theorem 4**: The minimum number of edges in a graph $G$ with $g$ points and having property $P_{3,1}$ is

\[
\begin{align*}
&\begin{cases}
  n^2 + 1 & \text{if } g = 2n + 1 \\
  n^2 - n + 1 & \text{if } g = 2n
\end{cases}
\end{align*}
\]

**Proof**: The proof will be by induction on the number of points in $G$. If $G$ has a cutpoint, then for property $P_{3,1}$ to hold, the two blocks determined by this cutpoint must be complete subgraphs. It is clear that the number of edges in $G$ would be at least the minimal number of edges in the dumbbell graph of $g$ points.

We now consider graphs with no cutpoints.

1) If $g = 4$, there are three edges in the smallest connected graph, and this graph has property $P_{3,1}$. If $g = 5$, there are two graphs with 5 edges that satisfy property $P_{3,1}$, and no graph with four edges that does. Five is the value expected from the formula.

![Figure 3](image)

2) Consider a graph $G$ with $g$ points. Remove one of
the points with maximal degree, \( v \). This point was incident with at most \( g-1 \) points of \( G \), so \( \deg(v) \leq g-1 \). Assume \( v \) was not joined to at least one point, \( w \), in \( G \). For property \( P_{3,1} \) to hold, \( v \) and \( w \), between them, must be joined to every point in \( G \).

\[
\deg(v) + \deg(w) \geq g-2
\]

Since \( v \) was a point of maximal degree, \( \deg(v) \geq \frac{g-2}{2} \). Now consider two cases.

Case 1: \( g = 2n + 1 \)

Now \( \deg(v) \geq \frac{2n+1}{2} - 1 \). Since \( \deg(v) \) is an integer, \( \deg(v) \geq n \), which means that \( v \) removed at least \( n \) edges from \( G \). By the inductive hypothesis, \( G_1 = G - \{v\} \) has \( n^2 - n + 1 \) edges, so \( G \) has at least \( n^2 - n + 1 + n = n^2 + 1 \) edges.

Case 2: \( g = 2n \)

Now \( \deg(v) \geq \frac{2n}{2} - 1 = n - 1 \). By the inductive hypothesis, \( G_1 = G - \{v\} \) has \( n^2 - 2n + 2 \) edges. Since \( v \) removed at least \( n-1 \) edges from \( G \), \( G \) has at least \( n^2 - 2n + 2 + n - 1 = n^2 - n + 1 \) edges.

**Definition:** A point \( g \) in \( G \) is star-complete if the subgraph \( G_1 \), generated by \( g \) and all points \( p_i \) such that \( d(g, p_i) = 1 \), is complete.

**Theorem 5:** Given a set of \( g \) points. A graph having property \( P_{3,1} \) can be generated with no star-complete points.

**Proof:**

Case 1: \( g \) is odd.
Choose one of the $g$ points to be the center $v_c$, and join each of the other points to this one. Next, number the points around the center $v_1, v_2, \ldots, v_{g-1}$ in order, and join $v_1$ to $v_2$, $v_3$ to $v_4, \ldots, v_{g-2}$ to $v_{g-1}$. Next, join $v_1$ to $v_3$, $v_2$ to $v_4$, $\ldots, v_{g-2}$ to $v_1$, $v_{g-1}$ to $v_2$. Next, join $v_1$ to $v_5$, $v_2$ to $v_6$, $\ldots, v_{g-2}$ to $v_2$, $v_{g-1}$ to $v_3$. Continue in this manner, until each point has $g-3$ edges incident to it.

This graph has property $P_{3,1}$, because any given point $v_i$ fails to be joined to only two other points, one next to it, and the one in position $v_{1+\frac{1}{2}(g-1)}$, and these two are joined.

No point is star-complete because every point is joined to one of the points next to it, and also to the point next to that. Those two points are not joined, so the point cannot generate a complete subgraph. The center point, $v_c$, cannot be star-complete because it is joined to every point, and thus the graph would have to be complete for $v_c$ to be star-complete.

![Figure 4]

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Case 2: \( g \) is even.

Again, designate one point the center, \( v_c \), and join all the other points to it. Number the points in order, and begin joining them in a manner similar to case 1, except that on the first round only, \( v_{g-1} \) is not joined to any other point. Continue until \( v_{g-1} \) has \( g-3 \) edges incident to it, and the other points have \( g-2 \) edges incident.

Property \( P_{3,1} \) follows because \( v_{g-1} \) fails to be joined to only two points, \( v_{g-2} \) and \( v_1 \), and these two are joined. All the other points in the graph fail to be joined to only one point.

The graph is not star-complete for the same reason as case 1.

Note: This procedure can be extended to property \( P_{3,2} \). For an odd number of points continue one more round of edges, making each point have \( g-2 \) edges incident to it. Then two edges exist between any three points, because each point is joined to every point except the one next to it. Any other point in the graph must join these two points. Star-completeness fails at each point for the same reason as in Theorem 4, above.

For an even number of points, join \( v_{g-1} \) to \( v_1 \). Now every point has \( g-2 \) or more edges incident, and \( P_{3,2} \) follows. No point is star-complete, because every point is joined to at least one pair of neighboring points which cannot be joined.

From Theorem 2 and Theorem 5, it follows that a graph can be generated having property \( P_{m,n} \), which has no star-complete points.
If time had allowed, I would have tried several other methods of characterizing graphs having property \( P_{m,n} \). One of the most interesting approaches would be through matrices. It is evident that a graph \( G \) satisfies property \( P_{3,1} \) if its adjacency matrix is such that at least one of the elements \( a_{ij}, a_{ik}, \) or \( a_{jk}, i \neq j \neq k \), is a one. It might be possible to extend this condition and to add some further qualifications, such as the rank of the matrix.

Another possible method of characterizing graphs having property \( P_{m,n} \) would be through connected spanning subgraphs; that is, a subgraph containing all the points of \( G \) and the minimum number of edges necessary for the subgraph to be connected. Guay [3] has shown that a connected graph \( G \), with \( g \) points, \( g \geq n \), whose points are star-complete except for some subset \( Q \) of \( n \) points, is spanned by the set of edges incident with the elements of \( Q \).

Identifying certain types of points within a graph might be another possible method of characterization. Star-complete points would be one type, which, when present in a certain combination, would mean that the graph has property \( P_{m,n} \).
References


