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Diffeomorphism invariance in general relativity

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Diffeomorphism Invariance in General Relativity

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1 Abstract

Einstein’s general relativity is a geometrical theory of gravity in which the effects of gravity are due to the curvature of space and time. In contrast, all of the other fundamental particle interactions are described as quantum field theories that are invariant under gauge transformations. In efforts to unify gravity with these other particle theories, it is desirable to reexpress Einstein’s general relativity as a gauge theory. The gauge symmetry in Einstein’s theory can be identified as invariance under diffeomorphism transformations. This work examines the nature of this symmetry, how it is implemented, and how it behaves as a gauge transformation. Since spontaneous symmetry breaking and the Higgs mechanism are important in particle physics gauge theories, these same mechanisms will be examined as well for a gauge theory containing diffeomorphism symmetry.

Using a fusion of well understood models from classical field theory and gauge theory, we investigate a gravity theory with a vector field that spontaneously breaks diffeomorphism symmetry. We conclude that spontaneous breaking, unlike conventional gauge theory, does not lead to the Higgs mechanism, and we examine the implications of this in greater detail.
2 Acknowledgements

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3 Introduction

Albert Einstein is perhaps best remembered as a genius for his developments in the theory of General Relativity (GR), and rightfully so. Einstein made several theoretical predictions for GR that have been consistently confirmed to this day. Furthermore, he developed the theory between the period of 1907 and 1915, which was long before the technology for experimental verification was available. The theory of General Relativity is a theory of gravitation that provides an accurate model of gravity that accounts for several unexplained problems in the classical Newtonian limit of gravity. GR describes gravitational attraction among masses as a curvature of spacetime, rather than an attractive force. A significant principle of GR is the principle of general covariance, which dictates that the fundamental laws of physics remain the same, regardless of coordinate frames. However, in the context of particle physics, this becomes a trivial assumption, as gauge theories in particle physics should not have a preferred frame to begin with.

Particle physics is the study of elementary particles and their interactions, and one of the most important results emerging from particle physics is the Standard Model. The Standard Model of particle physics is a theory which describes the electroweak and quantum chromodynamic interactions between particles. Therefore, the theory successfully unifies the strong, weak and electromagnetic interactions into one model, but it is missing the fourth fundamental interaction: gravity. The failure to successfully incorporate gravity into the Standard Model of physics outlines one of GR's greatest unsolved mysteries. Even though GR itself is consistent with experimental data, there are currently no theories that successfully incorporate quantum mechanics with the fundamental interaction of gravity. Since all of the other fundamental interactions are described as quantum field theories in the Standard Model, a good theoretical starting point to solving this problem is to reexpress GR as a gauge theory.

A gauge theory is a particle physics field theory that has a gauge transformation, or a mathematical transformation that can be performed and still leave the system unchanged. We can capture the physics of a system in gauge theories, where particles are expressed as scalar, vector and tensor fields. All gauge theories can be written in terms of a Lagrangian, which describes the dynamics of the system, and thus, gauge transformations can be thought of more specifically, as mathematical transformations that leave the Lagrangian of a gauge theory invariant. Since particle physics is derived without a preferred coordinate frame, any good gauge theory should require that the physics be the same in every frame, and thus, general covariance becomes trivial in the case of gauge transformations. Instead, we will see that the mathematically equivalent set of transformations known as diffeomorphisms are the relevant gauge transformation in GR.

As mentioned earlier, the Standard Model of particle physics lacks the inclusion of gravity as an interaction among particles. A better understanding of the mechanics of diffeomorphism invariance as a gauge symmetry in GR could lend clues as to how to incorporate gravity in other unifying models of physics. In this thesis, we will not take on the daunting task of trying to come up with a new unifying model of physics that includes gravity. Instead, we will examine the answers to the questions: Can GR be written as a gauge theory? If so, what is the relevant gauge transformation? I have already implied that GR can indeed be written as a gauge theory, where the diffeomorphisms are the relevant gauge transformation. We will investigate how this is possible in more detail and examine why GR is called a diffeomorphism invariant gauge theory.
Afterwards, we will look at a specific model, which is a simple model that could emerge from string theory, and aim to answer the following questions: Do features of the Standard Model of Particle Physics have analogues in our model? If analogues exist in our model, do they occur? In particular, these questions will examine the spontaneous symmetry breaking mechanism, which is a mechanism that occurs in the electroweak gauge theory of the Standard Model and provides some interesting results. This research will hopefully provide us with a better understanding of GR as a theory, and perhaps shed light on future research with the goal of finding a grand unifying model of physics that includes gravity.

In order to understand the intricacies of diffeomorphism invariance, we are required to form a new understanding of the mathematics of scalar, vector and tensor fields on manifolds. Therefore, in this thesis, we will deviate from the common understanding of a vector as an object with magnitude and direction, pointing from one location to another. Instead, we will embrace a more general definition of vectors in the context of manifolds (see Glossary in sec. [10] for definitions), where vectors are objects that operate on the space of smooth functions in a manifold. Also, GR utilizes heavy mathematical machinery to describe the dynamics of vectors and tensors, and to conveniently keep the notation concise, GR uses the Einstein summation convention. This thesis will assume that the reader is competent in working in this convention. Also, we will assume the convention that the flat space Minkowski metric is

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and we will assume that $c = \hbar = 1$, where $c$ is the speed of light and $\hbar$ is Planck's constant.

This thesis will begin by going through the physics behind field theory, gauge invariance, diffeomorphisms, and the spontaneous symmetry breaking mechanism which arises in the Standard Model. Once this background has been established, we will examine a specific model with a vector field, called the Bumblebee Model, that looks at the effects of spontaneously breaking the diffeomorphism invariance of the theory. To begin, we start by examining classical Lagrangian field theory.

## 4 Classical Lagrangian Field Theory

In this section, we examine classical field theory and the methods used in finding the equations of motion for a given model. All models in classical field theory can be expressed in terms of a Lagrangian. The Lagrangian is a function that describes the dynamics of a system and takes the form

$$\mathcal{L} = T - V,$$

where $T$ and $V$ are the kinetic and potential energy of the system, respectively. The integral of the Lagrangian is called the action, that is,

$$S = \int \mathcal{L} \, dt \, dz.$$
The action possesses the special property that small variations in the action should equal 0, or that
\[ \delta S = \int \delta \mathcal{L} \, d^4x = 0, \]  
where \( \delta S \) is the variation of the action. This property is known as the principle of least action.

The idea behind Lagrangian field theory is that, we can describe particles as fields in a Lagrangian and thus small variations in the Lagrangian should be equal to zero by the principle of least action. Therefore, by varying the fields in a Lagrangian and using the principle of least action, we can solve for the particle's equations of motion.

4.1 Classical Scalar Field

As a first example, consider a theory with a massive scalar field \( \phi \). The Lagrangian associated with the field is
\[ \mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2, \]  
where \( \phi \) is the scalar field, \( m \) is the mass and \( \partial_{\mu} \) represents
\[ \partial_{\mu} = \frac{\partial}{\partial x^\mu}. \]

We can solve for the theory's equations of motion by using the principle of least action. If we vary (5) with respect to \( \phi \), such that \( \phi \rightarrow \delta \phi \), we get
\[ \delta \mathcal{L} = \frac{1}{2} (\partial_{\mu} \delta \phi)(\partial^\mu \phi) + \frac{1}{2} (\partial_{\mu} \phi)(\partial^\mu \delta \phi) - m^2 \phi \delta \phi. \]  
We know that we will eventually want the variation of the action to be equal to 0. From the definition of the action, we know that \( \delta \mathcal{L} \) will appear in the integral, and therefore we can use integration by parts on (7). Integration by parts on the first term of (7) would give us
\[ \int \frac{1}{2} (\partial_{\mu} \delta \phi)(\partial^\mu \phi) = \int \partial_{\mu} \left[ \frac{1}{2} \delta \phi \partial^\mu \phi \right] - \int \frac{1}{2} \delta \phi \partial_{\mu} \partial^\mu \phi \]
\[ = 0 - \int -\frac{1}{2} \delta \phi (\partial_{\mu} \partial^\mu \phi). \]  
The first term vanishes since the endpoints do not change in small variations, and thus, \( \delta \phi \) evaluated at the endpoints is equal to 0. We can extend this concept of integration by parts to any term we encounter where the \( \delta \phi \) is nested inside of a derivative \( \partial_{\mu} \), and we will always be able to drop the total derivatives since \( \delta \phi \) will always equal 0 when evaluated at the endpoints. Therefore, (7) reduces to
\[ \delta \mathcal{L} = -\frac{1}{2} \delta \phi (\partial_{\mu} \partial^\mu \phi) - \frac{1}{2} (\partial^\mu \partial_{\mu} \phi) \delta \phi - m^2 \phi \delta \phi. \]  
Using the fact that in summation convention, \( \partial_{\mu} \partial^\mu = \partial^\mu \partial_{\mu} \), we can rewrite (9) as
\[ \delta \mathcal{L} = -\delta \phi [\partial_{\mu} \partial^\mu \phi - m^2 \phi]. \]
Eq. (10) must be equal to zero by the least action principle. Therefore, we get that

$$\Box \phi + m^2 \phi = 0,$$

where $\Box$ is called the D'Alembertian operator, defined as

$$\Box = \partial_\mu \partial^\mu.$$  

Eq. (11) gives the equations of motion for the classical scalar field, $\phi$, defined in the Lagrangian of (5). Solutions to (11) are of the form

$$\phi = e^{-ik_\mu x^\mu},$$

where

$$k_\mu k^\mu = m^2.$$  

The solution of (13) can be verified by plugging it back in to the equations of motion. We won't actually do the substitution here, but if we did, we would verify that (14) must necessarily be true for the solution to hold. The important point to note here is the conceptual meaning of the solution and its restriction in (14).

By definition, we have that

$$p^\mu = \hbar k^\mu = \frac{E}{c}, \vec{p}.$$  

Recall that our conventions are $c = \hbar = 1$, and thus,

$$k^\mu = (E, \vec{p}),$$

or equivalently,

$$k^0 = E, \quad \vec{k} = \vec{p}.$$  

Thus, we can manipulate (14) to show that

$$k^\mu k_\mu = m^2$$

$$\frac{(k^0)^2}{k^2} - \frac{\vec{E}^2}{m^2} = m^2$$

$$\frac{(k^0)^2}{m^2} = m^2 + \frac{\vec{E}^2}{\vec{p}^2},$$

where $E$ is the energy of the particle, and $\vec{p}$ is its momentum vector. Eq. (18) is precisely the relation a massive scalar should obey! Thus, we have just shown that solutions for $\phi$ that are of the form (13) with the restriction (14) imply a massive free particle described as a plane wave with mass $m$.

Therefore, we can make the conclusion that whenever we see the two terms of (5) in any part of a given Lagrangian, we know that Lagrangian theory involves a massive scalar field in its solution. This will assist in what to expect in the solutions for more complicated Lagrangians, such as in the electromagnetic case.
4.2 E and M Field

Here we look at the field theory associated with electromagnetism. We have the Lagrangian field

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \]  

(19)

where \( F_{\mu\nu} \) is the strength tensor, which is a rank 2 tensor that includes the components for the associated electric and magnetic fields,

\[ F_{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}. \]  

(20)

Mathematically, it is defined as

\[ F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]

(21)

with \( A_\mu \) as the 4-vector potential field. Here in the EM case, the \( A_\mu \) vector is the analogue to the scalar field \( \phi \) from the previous case. In the previous case, we worked with scalar fields, and in the EM case here we are working with \( F_{\mu\nu} \), which contains the vector field \( A_\mu \).

We want to get our Lagrangian in terms of \( A_\mu \) before we can perform any variations on it. If we substitute (21) into (19), then the expanded Lagrangian becomes

\[ \mathcal{L} = -\frac{1}{4} \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) \left( \partial^\mu A^\nu - \partial^\nu A^\mu \right) \]

\[ = -\frac{1}{4} \left[ \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\mu - \partial_\nu A_\mu \partial^\nu A^\mu + \partial_\mu A_\nu \partial^\nu A^\mu \right] \]

\[ = \frac{1}{4} \left[ 2 \partial_\mu A_\nu \partial^\mu A^\nu - 2 \partial_\mu A_\nu \partial^\nu A^\mu \right]. \]  

(22)

Now that we have the Lagrangian entirely in terms of \( A_\mu \), we can solve for its equations of motion using the principle of least action. If we vary (22) with respect to \( A_\mu \), such that \( A_\mu \rightarrow \delta A_\mu \), then we get

\[ \delta \mathcal{L} = -\frac{1}{2} \partial_\nu \delta A_\mu \partial^\nu A^\mu - \frac{1}{2} \partial_\nu \delta A_\mu \partial^\nu A^\mu + \frac{1}{2} \partial_\nu \delta A_\mu \partial^\nu A^\mu + \frac{1}{2} \partial_\nu A_\mu \partial^\nu \delta A^\mu \]

\[ = -\partial_\nu \delta A_\mu \partial^\nu A^\mu + \partial_\nu \delta A_\mu \partial^\nu A^\mu. \]

(23)

We use integration by parts to get

\[ \delta \mathcal{L} = \delta A_\mu \partial_\nu \partial^\nu A^\mu - \delta A_\nu \partial_\mu \partial^\nu A^\mu \]

\[ = \delta A_\nu [\partial_\mu \partial^\nu A^\mu - \partial_\nu \partial^\nu A^\mu], \]

(24)

and therefore the equations of motion in the EM case are

\[ \Box A^\nu - \partial_\nu \partial^\nu A^\mu = 0. \]  

(25)

These equations of motion give Maxwell's equations. We can make a couple observations immediately from these equations of motion. Unlike the classical scalar case, we do not have a
mass term in the equations of motion, and thus, any propagating solutions would have to represent photons. Furthermore, we can note that in the $\nu = 0$ equation, the second time derivatives on $A_0$ cancel out. For this reason, $A_0$ is not a physical propagating field and is known as an auxiliary field.

For the 3 remaining degrees of freedom there should be 3 possible solutions, but as it turns out, only 2 will represent independent physical degrees of freedom. How can one reduce the 3 degrees of freedom down to just 2 physical solutions? It turns out that the EM theory has a redundant gauge degree of freedom as a result of gauge invariance, that is to say, the EM theory has a particular transformation on $A_\mu$ that leaves its Lagrangian invariant. Let’s look at the concept of gauge invariance to see this in more detail. I will first show that the EM theory has a gauge symmetry, and then I will show how that gauge symmetry can be fixed to reduce our solution down to 2 physical degrees of freedom.

Recall that gauge invariance results from a field theory having a gauge transformation, or a mathematical transformation that leaves the Lagrangian invariant. In the EM case, the gauge transformation is a 4-vector transformation on $A_\mu$, defined as

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda,$$  

(26)

where $\Lambda$ is an arbitrary scalar function such that $\partial_\mu \Lambda$ changes $A_\mu$ infinitesimally. $\Lambda$ is sometimes referred to as the gauge parameter because it is a parameter in the gauge transformation that we have the freedom to arbitrarily define. This transformation is unique because if we apply (26) to $F_{\mu\nu}$, we see that this transformation leaves $F_{\mu\nu}$ unchanged. More explicitly,

$$F_{\mu\nu} \rightarrow \partial_\nu(A_\mu + \partial_\mu \Lambda) - \partial_\mu(A_\nu + \partial_\nu \Lambda)$$

$$\rightarrow \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda$$

$$\rightarrow F_{\mu\nu}.$$  

(27)

By similar derivation, the transformation leaves $F^{\mu\nu}$ unchanged as well, and thus, it is easily verified that the 4-vector gauge transformation leaves the EM Lagrangian invariant!

How can the gauge symmetry from the 4-vector transformation of (26) be used to eliminate a redundant degree of freedom? Recall that the $F_{\mu\nu}$ tensor describes the EM fields. Therefore, the conceptual interpretation of this mathematical symmetry is that the function, $\Lambda$, can be arbitrarily chosen and we would still be left with the same electromagnetic fields, $F_{\mu\nu}$. The freedom to choose any gauge parameter is called gauge invariance. The act of choosing a specific $\Lambda$ is referred to as fixing the gauge. For the gauge transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda,$$  

(28)

we can effectively eliminate one of the possible degrees of freedom by fixing the gauge. In the EM case, a common choice of gauge is

$$\partial_\mu A^\mu = 0.$$  

(29)

To be specific, we pick a $\Lambda$ such that

$$\partial_\mu A^\mu + \partial_\mu \partial_\nu \Lambda = 0$$

$$\partial_\mu \partial^\mu \Lambda = -\partial_\mu A^\mu$$

$$\Box \Lambda = -\partial_\mu A^\mu.$$  

(30)
So now in our gauge-fixed theory, we have what is called the Lorentz condition

\[ \partial_{\mu} A^\mu = 0. \]  

When (31) is applied to the previous equations of motion in (25), we are left with just

\[ \Box A_\mu = 0 \]  

to describe the 2 physical degrees of freedom. Solving for the solutions gives 2 physically propagating modes, which are the transverse massless photon modes.

Therefore, to summarize the conclusions of the EM case, a Lagrangian in the form of (19) results in 2 transverse massless photon modes. This seems to disagree with the unaugmented equations of motion we find, which would suggest that there are 4 potential solutions, however, one mode is a non-propagating auxiliary mode and another is a redundant degree of freedom that results from the theory's inherent gauge invariance. This gauge degree of freedom can be eliminated by fixing the gauge. Finally, note that the gauge invariance lends further evidence that the solutions to the EM theory must be massless. This is because adding a mass term \( m^2 A_\mu A^\mu \) to the Lagrangian would destroy this gauge invariance, and therefore, if gauge invariance is a requirement of the theory, the gauge fields (i.e. the photons) must remain massless. We encounter similar methods of using the properties of gauge invariance to eliminate redundant degrees of freedom in the GR case.

4.3 GR Free Space (Einstein-Hilbert Action)

In this section we formulate GR as an action principle and derive Einstein's equations from the Lagrangian field variation. We will find that, similar to the EM case, there is a gauge symmetry of the theory, and it can be used to eliminate redundant degrees of freedom in the equations of motion. We will start with the Lagrangian

\[ L = \sqrt{-g} R, \]  

where \( g = |g_{\mu\nu}| \) and \( g_{\mu\nu} \) is the metric tensor, which is a rank 2 tensor that describes the curvature of the associated spacetime. \( R \) is the Ricci tensor, defined as

\[ R = R^\mu_{\mu} = g^{\rho\sigma} R_{\rho\sigma}, \]  

where \( R_{\rho\sigma} \) is the contraction

\[ R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}, \]  

where \( R^\lambda_{\mu\lambda\nu} \) is the Riemann curvature tensor, defined as

\[ R^\lambda_{\rho\lambda\nu} = \partial_\rho \Gamma^\lambda_{\lambda\nu} - \partial_\nu \Gamma^\lambda_{\rho\lambda} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\mu}, \]  

where \( \Gamma \) is the Christoffel symbol, defined as

\[ \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \]  

Therefore, in its most primitive form, we see that the GR Lagrangian is a function of the \( g_{\mu\nu} \) tensor, or that \( L = L(g_{\mu\nu}) \).
Here in the GR case, the $g_{\mu\nu}$ tensor is the analogue to the $A_\mu$ vector from the EM case. Therefore, variations in the metric tensor will lead us to the equations of motion of the theory. We will choose to vary with respect to $g^{\mu\nu}$, such that $g^{\mu\nu} \to \delta g^{\mu\nu}$. Equivalently, we can define variations in $\delta g_{\mu\nu}$ as

$$\delta g_{\mu\nu} = -g_{\mu\sigma}g_{\nu\tau}\delta g^{\sigma\tau}. \quad (38)$$

It should be noted that choosing to vary with respect to $g^{\mu\nu}$ is simply a common convention and we could just have easily chosen to vary with respect to $g_{\mu\nu}$ and derive $\delta g^{\mu\nu}$.

If we vary (33) with respect to $g^{\mu\nu}$, we get

$$\delta \mathcal{L} = (\delta \sqrt{-g})g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}(\delta g^{\mu\nu})R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}(\delta R_{\mu\nu}) \quad (39)$$

With some extensive work involving $\Gamma$ identities, we can solve for $\delta R_{\mu\nu}$ in terms of $\delta g^{\mu\nu}$, see Ref. [2], where we find

$$\delta R_{\mu\nu} = 0. \quad (40)$$

This just leaves the first term of (39) left to determine since the second term is fine as it is, because it is already in terms of $\delta g^{\mu\nu}$. To figure out what $\delta \sqrt{-g}$ is in terms of $\delta g^{\mu\nu}$, we need to do some work. Using the rule that, for a matrix $M$

$$\ln(\text{det}M) = \text{Tr}(\ln M), \quad (41)$$

we can arrive at the conclusion that

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} = -g g_{\mu\nu}\delta g^{\mu\nu}. \quad (42)$$

Then, taking the square root,

$$\delta \sqrt{-g} = \frac{1}{2}(-g)^{-\frac{1}{2}}[-\delta g]. \quad (43)$$

Plugging (42) into (43), we get

$$\delta \sqrt{-g} = \frac{1}{2}\frac{1}{\sqrt{-g}} [-g g_{\mu\nu}\delta g^{\mu\nu}]$$

$$= \frac{1}{2} \frac{-g^{\mu\nu}}{\sqrt{-g}} g_{\mu\nu}\delta g^{\mu\nu} \quad (44)$$

and therefore we get the result we want

$$\delta \sqrt{-g} = \frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}. \quad (45)$$

If we substitute (40) and (45) into (39), we are left with

$$\delta \mathcal{L} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}(\delta g^{\mu\nu})R_{\mu\nu} + \sqrt{-g}(\delta g^{\mu\nu})R_{\mu\nu}$$

$$= \sqrt{-g} \left[ -\frac{1}{2}g_{\mu\nu}(\delta g^{\mu\nu})R_{\mu\nu} + (\delta g^{\mu\nu})R_{\mu\nu} \right]$$

$$= \sqrt{-g} \left[ -\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu} \right] \delta g^{\mu\nu}. \quad (46)$$
We use the principle of least action to solve for the theory's equations of motion. Eq. (46) must be equal to 0. Since $\sqrt{-g} \neq 0$, it must be that

$$G_{\mu\nu} = 0,$$

(47)

where $G_{\mu\nu}$ is called the Einstein tensor, defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R.$$

(48)

This gives the equations of motion for the GR free space case.

The solutions to these equations of motion are freely propagating gravity waves, otherwise referred to as gravitons. To study these solutions, we must consider $g_{\mu\nu}$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

(49)

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is some small perturbation. The gravity waves are then in the $h_{\mu\nu}$ fields, and therefore, it is sufficient to linearize the theory to have its equations of motion to be in terms of $h_{\mu\nu}$. After a lot of grunt work, see Ref. [2], we get

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}h_{\mu\nu}R = \frac{1}{2}(\partial_\alpha \partial_\nu h^{\sigma\mu} + \partial_\sigma \partial_\mu h^{\alpha\nu} - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial_\rho \partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu} \Box h),$$

(50)

where $G_{\mu\nu} = 0$ and $h$ is the contraction

$$h = \eta^{\mu\nu}h_{\mu\nu} = h_\mu^\mu.$$

(51)

Despite its complex appearances, (50) will eventually end up with only 6 potential independent solutions. How is this possible, when $h_{\mu\nu}$ is a two-index tensor? For starters, the fact that $h_{\mu\nu}$ is symmetric, reduces our equations of motion from 16 down to 10 independent solutions. Furthermore, as was true for the case in the EM solutions, the GR theory has a gauge invariance and we can eliminate redundant degrees of freedom by fixing the gauge.

The gauge symmetry in GR is the diffeomorphism, which will be described in more detail in section (5). It will be shown that when these gauge transformations are applied to $h_{\mu\nu}$, they take the form

$$h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu,$$

(52)

where $\xi_\mu$ is a gauge parameter with 4 gauge degrees of freedom due to its $\mu$ subscript. Analogously to the EM case, this gauge transformation leaves Einstein's equations invariant. This gauge can be fixed with the standard choice of gauge as

$$\partial_\alpha h^{\alpha\mu} = 0,$$

(53)

where we've defined

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}.$$

(54)
In the gauge-fixed theory, Einstein's equations reduce to

$$\Box \tilde{h}_{\mu\nu} = 0$$

(55)

with condition (53) in place. Since (53) is actually 4 separate conditions, the act of fixing the gauge effectively eliminates 4 redundant degrees of freedom, moving us from 10 down to 6 independent solutions. Then when we examine the solutions to the equations of motion we find that there are auxiliary modes in Einstein's GR, similar to what we saw in the EM case. In solving the theory, we would find that 4 of the solutions are non-propagating modes, and therefore we are only actually left with 2 physical gravity waves in the final solution.

Therefore, to summarize the conclusions of the GR free space theory, a Lagrangian in the form of (33) results in 2 physical gravity waves, known as gravitons. The ungauged equations of motion would initially suggest that there are 16 potential solutions, however, symmetry in the metric reduces the independent solutions down to 10. Then, 4 redundant degrees of freedom can be gauged away, leaving 6 independent degrees of freedom. Finally, solving the equations of motion in the gauge-fixed theory shows that there are only 2 physically propagating modes, where the other 4 are auxiliary modes. It is important to note that the entire theory here is Einstein's equations in free space, meaning there is no matter here, only curvature. This is actually a very simplified case. If we wanted to generalize, we would add matter into the theory by adding other particle fields to the Lagrangian. These matter fields would be summarized in the form of an energy-momentum tensor $T_{\mu\nu}$.

5 Diffeomorphism Invariance in GR

As a consequence of gauge invariance, our EM Lagrangian from (19) was left unchanged under the 4-vector EM transformation from (26). For GR, we will see that diffeomorphisms are the transformations analogous to (26) that leave the GR Lagrangian invariant. Therefore, General Relativity is often referred to as a diffeomorphism invariant theory, in which the diffeomorphism acts as a gauge transformation. In this section, we will examine the mathematical definition of the symmetry and see how it acts on tensor and vector fields.

5.1 Active and Passive transformations in GR

When talking about transformations, it is important to distinguish between active and passive transformations. An active transformation is generally defined in the absence of a coordinate system and changes the physical state of the system, but a passive transformation is simply a change of coordinates, and thus takes on no physical meaning. A basic example of the difference between active and passive transformations is manifested when trying to rotate a vector. We can rotate the vector by grabbing it and turning it by $\theta$ degrees (active), or we can redefine our coordinates by rotating our axis by $\theta$ degrees (passive).

In GR, the pertinent active transformation is a diffeomorphism, which is a mapping between manifolds. The equivalent passive transformation would be a general coordinate transformation. As stated earlier, active and passive transformations are mathematically equivalent in theories where
neither of these symmetries are broken. But in the view of particle physics and gauge theories, every theory should be coordinate independent and thus, the passive transformation is trivial. This is to say, that we will focus on the effects of the active transformation of diffeomorphisms on our gauge theories, rather than considering passive coordinate transformations. We will also ultimately consider the possibility that these symmetries are spontaneously broken.

5.2 Manifolds

In order to talk about diffeomorphisms and curved space, we must introduce the concept of manifolds. Manifolds are spaces which may be complicated and hard to visualize, but when we zoom in, the space resembles n-dimensional Euclidean space, or $\mathbb{R}^n$. This is achieved by constructing the manifold out of several local regions which are smoothly sewn together.

If we want to be mathematically rigorous about our definition, we must discuss the notion of a mapping between manifolds. Informally, a mapping is a generalization of a function. More formally, given two manifolds, $M$ and $N$, a map $\varphi : M \to N$ is a relationship that assigns each element of $M$ to exactly one element of $N$.

The definitions of vectors and tensors also deviate from our common understanding when we are thinking of them in the context of a manifold. Vectors and tensors are quantities that act as operators on the manifold. Given some manifold $M$, vectors are objects defined at a specific point $p \in M$ that operate on the space of all smooth functions at that point $p$. The associated components of the vector, responsible for defining its magnitude and direction, are determined by how the vector operates on the space of all smooth functions at that point $p$. Mathematically, for a function $f$ and vector $V$,

$$ V : f \to \mathbb{R}^n. \quad (56) $$

Figure 1: Comparison of active vs. passive transformation for rotation of a vector.
Likewise, we can define dual vectors which are associated with the manifold's dual space, or cotangent space. A dual vector is an object that operates on vectors at a specific point in the manifold. Mathematically, for a dual vector $W$ and a vector $V$,

$$W : V \rightarrow \mathbb{R}^n. \quad (57)$$

Tensors are simply the generalization of vectors and dual vectors, in the sense that, a tensor can contain several contravariant and covariant components, which by themselves act as vectors and dual vectors. Tensors have the property that they transform properly under general coordinate transformations, or that

$$T^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_l} = \frac{\partial x^{\mu_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x^{\mu_k}}{\partial x^{\nu_l}} T^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_l}. \quad (58)$$

Now that we have defined how vectors and tensors operate in manifolds, it is convenient to discuss the concept of mappings between manifolds. In this discussion, the pullback and pushforward of a function are useful concepts. Say we have two manifolds, $M$ and $N$, and suppose that there exists a mapping, $\phi : M \rightarrow N$, and a function $f : N \rightarrow \mathbb{R}$. The pullback of $f$, written as $\phi^* f$, is simply the composition of $f$ with $\phi$, or in mathematical terms,

$$\phi^* f = (f \circ \phi). \quad (59)$$

In effect, it seems as if the function $f$ has been pulled back from the manifold $N$ to the manifold $M$, so it would appear that we have the capability of operating $f$ on manifold $M$.

![Figure 2: Chart showing the functional relations between the manifolds $M$ and $N$ (figure recreated from Ref. [2]). $\phi$ is a mapping from $M$ to $N$ and $f$ is a function from $N$ to $\mathbb{R}$. The pullback is the composition $(f \circ \phi)$.](image)

Keeping our definitions of $\phi$ and $f$, say we had a function $g : M \rightarrow \mathbb{R}$ and we wanted to push it forward from the manifold $M$ to the manifold $N$, so it would appear that we have the capability of operating $g$ on manifold $N$. There is a slight problem here in that it is impossible to create a function on $N$ as a result of some composition between the $g$ and the mapping $\phi$. Compositions simply cannot accomplish this task. But recall that a vector can be thought of as a derivative operator that maps smooth functions to real numbers. Therefore, we can think of pushing forward a vector in order to achieve the functionality of $g$ operating on the manifold $N$. This process is called
the pushforward of a vector, where the vector acts as a derivative operator on functions such that for a vector \( V \) and a function \( f \) on the manifold, \( V : f \rightarrow \mathbb{R}^n \). If \( V(p) \) is a vector at point \( p \) on manifold \( M \), then the pushforward vector, written as \( \phi_* V \), at the point \( \phi(p) \) on the manifold \( N \) is given by its action on the functions on \( N \), written as \( F(N) \),

\[
(\phi_* V(p))(F(N)) = V(p)(\phi^* F(N)).
\]  

Creating a functional chart, we would see

\[
\begin{array}{c}
\mathbb{R} \\
\mid \downarrow \phi_* \\
V(p) \\
\phi_* (V(p)) = V(p) \circ \phi^* \\
\mid \downarrow \phi^* \\
F(M) \\
F(N)
\end{array}
\]

Figure 3: Chart showing the functional relations for a pushforward of a vector at point \( p \) on manifold \( M \), \( V(p) \) (recreated from Ref. [2]). \( F(N) \) and \( F(M) \) are the space of functions on the manifold \( N \) and \( M \), respectively.

From the definitions of pullbacks and pushforwards, we can see that mappings can be used to pull certain things back and push other things forward. We won’t dive too deeply in the discussion of the two, but in the end, we find that we can pull back tensors with an arbitrary number of lower indices and that we can push forward tensors with an arbitrary number of upper indices. Mathematically speaking, for a \((0, j)\) tensor \( T_{\alpha_1 \ldots \alpha_j} \), the pullback of \( T \) is

\[
(\phi^* T)_{\mu_1 \ldots \mu_j} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \ldots \frac{\partial y^{\alpha_j}}{\partial x^{\mu_j}} T_{\alpha_1 \ldots \alpha_j}.
\]  

Likewise, for a \((k, 0)\) tensor \( S^{\mu_1 \ldots \mu_k} \), the pushforward of \( S \) is

\[
(\phi_* S)_{\alpha_1 \ldots \alpha_k} = \frac{\partial y^\alpha}{\partial x^{\mu_1}} \ldots \frac{\partial y^\alpha}{\partial x^{\mu_k}} S^{\mu_1 \ldots \mu_k}.
\]  

A chart showing the allowed pullback and pushforward operations for tensors on manifolds \( M \) and \( N \) is shown in the figure on the next page.

We will see in the next section that our understanding of the way vectors and tensors work on a manifold, along with our definitions of a pullback and pushforward of a tensor, will help define a diffeomorphism.
5.3 Diffeomorphisms

With the background on mappings, pullbacks, and pushforwards, we can now discuss what a diffeomorphism is. For two given manifolds, \( M \) and \( N \), a \textbf{diffeomorphism} is an invertible mapping \( \phi : M \rightarrow N \), such that \( \phi \) and \( \phi^{-1} \) are both \( C^\infty \) mappings. Unfortunately, the concept of having two abstract manifolds is a very mathematical understanding of a diffeomorphism and it is difficult to apply to the field of physics. In the context of physics, we are working with the spacetime manifold and thus we can think of a diffeomorphism occurring on the same manifold, that is, we can define diffeomorphism as mappings \( \phi : M \rightarrow M \).

If we have a mapping \( \phi : M \rightarrow M \), then we see that \( \phi \) is obviously invertible. Provided that \( \phi \) is a \( C^\infty \) mapping, then \( \phi \) is automatically a diffeomorphism. What is also convenient about \( \phi : M \rightarrow M \) mappings is that now we can \textit{pullback} and \textit{pushforward} the same thing, whether it is a scalar, vector, or an arbitrary mixed tensor. This is done by using the inverse mapping \( \phi^{-1} \). More specifically, the pushforward of a tensor under \( \phi \) is given by the pullback under \( \phi^{-1} \). This convenience allows us to define how tensors change under diffeomorphism (see section (5.4) below).

Carlo Rovelli, in Ref. [7], provides a very good conceptual example of a diffeomorphism in his book \textit{Quantum Gravity}. "Consider the surface of the Earth as a manifold, and call it \( M \). At each point \( P \in M \) on Earth, say the city of Paris, there is a certain temperature \( T(P) \). The temperature is a scalar function \( T : M \rightarrow R \) on the Earth's surface. Imagine a simplified model of weather evolution in which the only factor determining temperature change was the displacement of air due to wind. By this I mean the following. Fix a time interval: say we call \( T \) the temperature on May 1st, and \( \bar{T} \) the temperature on May 2nd. During this time interval, the winds move the air which is over a point \( Q = \phi(P) \) to the point \( P \). If, say, \( Q \) is the French village of Quintin, this means that the winds have blown the air of Quintin to Paris. Assume the temperature \( \bar{T}(P) \) of Paris on May 2nd is equal to the temperature \( T(Q) \) of Quintin the day before. The "wind" map \( \phi \) is a map from the Earth's surface to itself, which associates with each point \( P \) the point \( Q \) from which the air has been blown by the wind. Assuming it is \textit{smooth} and \textit{invertible}, the map \( \phi : M \rightarrow M \) is an active diffeomorphism." By smooth mapping, we mean a mapping that is continuous and infinitely differentiable, and by invertible mapping, we mean a mapping whose inverse still acts as a function.
With the definition and conceptual background of a diffeomorphism, we can see how scalar, vector and tensor fields change under diffeomorphisms by examining the concept of Lie derivatives.

5.4 Lie derivatives

The concept of derivatives measuring a rate of change becomes more complicated when working on a manifold. As mentioned earlier, when a tensor is moved from point $P$ to point $Q$ as a result of a diffeomorphism, the tensor must be pulled back to its original location $P$ and be compared to its original orientation. The Lie derivative can be used to effectively measure this change.

A Lie derivative is a derivative which measures change of a quantity on a manifold. Consider a diffeomorphism mapping where $X^\mu \rightarrow X^\mu + \xi^\mu$, where $\xi^\mu$ is an infinitesimal change on the manifold. Then, the change in a scalar, vector, or tensor field $U$ is given by the Lie derivative, denoted by $\mathcal{L}_\xi U$. An important notational convention for this thesis is that $\mathcal{L}$ by itself will be used to represent the Lagrangian, whereas $\mathcal{L}_\xi$ with a subscript will represent a Lie derivative with respect to a field $\xi$.

The Lie derivative for a scalar field $\phi$, with respect to a field $\xi$ is given by

$$\mathcal{L}_\xi \phi = \xi^\mu \partial_\mu \phi.$$  \hfill (63)

We can extend the concept of Lie derivatives for a vector field. The Lie derivative for a vector field can be found by calculating what is called the Lie Bracket. Consider two vector fields $X$ and $Y$, such that

$$X = X^\mu \partial_\mu,$$  

$$Y = Y^\mu \partial_\mu,$$  \hfill (64)

where $\partial_\mu$ form a basis set. The Lie bracket is the commutator given by

$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu.$$  \hfill (65)

As mentioned above, the Lie derivative of a vector field $V^\mu(x)$, along the vector field $\xi$ is given by the Lie bracket $[\xi, V]$

$$\mathcal{L}_\xi V^\mu = [\xi, V]^\mu.$$  \hfill (66)

This idea of a Lie derivative of a vector field can be generalized to a tensor field definition. For an arbitrary tensor of the form $T^{\mu_1 \mu_2 \ldots \mu_n}{}_{\nu_1 \nu_2 \ldots \nu_l}$,

$$\mathcal{L}_\xi T^{\mu_1 \mu_2 \ldots \mu_n}{}_{\nu_1 \nu_2 \ldots \nu_l} = V^\sigma \partial_\sigma T^{\mu_1 \mu_2 \ldots \mu_n}{}_{\nu_1 \nu_2 \ldots \nu_l}$$

$$- (\partial_\lambda V^\mu_1) T^{\lambda \mu_2 \ldots \mu_n}{}_{\nu_1 \nu_2 \ldots \nu_l}$$

$$- (\partial_\lambda V^\mu_2) T^{\mu_1 \lambda \ldots \mu_n}{}_{\nu_1 \nu_2 \ldots \nu_l} - \ldots$$

$$+ (\partial_\nu_1 V^\lambda) T^{\mu_1 \mu_2 \ldots \mu_n}{}_{\lambda \nu_2 \ldots \nu_l}$$

$$+ (\partial_\nu_2 V^\lambda) T^{\mu_1 \mu_2 \ldots \mu_n}{}_{\nu_1 \lambda \ldots \nu_l} + \ldots$$  \hfill (67)

We will use this definition to show how tensors transform under diffeomorphisms.
5.5 Diffeomorphism Invariance

In this subsection, we will show by brute force, that GR is diffeomorphism invariant. Let's begin by defining a diffeomorphism

\[ X^\mu \rightarrow X^\mu + \xi^\mu. \]  

(68)

For a vector field, the change \( \delta A_\mu \) under the diffeomorphism will be given by the Lie derivative

\[ \delta A_\mu = \mathcal{L}_\xi A_\mu = -\xi^\sigma \partial_\sigma A_\mu - (\partial_\mu \xi^\lambda) A_\lambda. \]  

(69)

Thus, the diffeomorphism transformations for \( A_\mu \) are

\[ A_\mu \rightarrow A_\mu - \xi^\sigma \partial_\sigma A_\mu - (\partial_\mu \xi^\lambda) A_\lambda, \]  

(70)

and similarly for \( A^\mu \)

\[ A^\mu \rightarrow A^\mu - \xi^\sigma \partial_\sigma A^\mu + (\partial_\lambda \xi^\mu) A^\lambda. \]  

(71)

Using the Lie derivative definition for a generic 2-term tensor, \( \mathcal{L}_\xi T_{\mu\nu} \), we get the diffeomorphism transformation for \( g_{\mu\nu} \) as

\[ g_{\mu\nu} \rightarrow g_{\mu\nu} - \xi^\sigma \partial_\sigma g_{\mu\nu} - \partial_\mu \xi^\lambda g_{\lambda\nu} - \partial_\nu \xi^\lambda g_{\mu\lambda}, \]  

(72)

and similarly for \( g^{\mu\nu} \)

\[ g^{\mu\nu} \rightarrow g^{\mu\nu} - \xi^\sigma \partial_\sigma g^{\mu\nu} + \partial_\mu \xi^\lambda g^{\lambda\nu} + \partial_\nu \xi^\lambda g^{\mu\lambda}. \]  

(73)

With our diffeomorphism transformations defined, let's look at the Einstein-Maxwell theory. This theory has a GR piece and an EM piece in its Lagrangian,

\[ \mathcal{L} = \frac{1}{16\pi G} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \]  

(74)

In this notation, the action is

\[ S = \int \sqrt{-g} \mathcal{L} d^4x, \]  

(75)

and \( S \) will be invariant provided that \( \mathcal{L} \) transforms as a scalar. In the previous section, we saw how scalars transform under Lie derivatives,

\[ \mathcal{L} \rightarrow \mathcal{L} - \xi^\sigma \partial_\sigma \mathcal{L}. \]  

(76)

To show that the Einstein-Maxwell Lagrangian transforms properly under diffeomorphism, we need to show that each of its pieces transforms as a scalar, as defined in (76) above. By itself, the first term is simply the Hilbert action, which we showed to be a legitimate and transformable Lagrangian in section (4.3). Therefore, we will take it for granted that the first term of (74) transforms properly, as defined by (76).
We have yet to show that the second term transforms properly. Let's first determine how the tensor $F_{J\nu}$ would transform under our diffeomorphism. It transforms as

$$F_{J\nu} \rightarrow F_{J'\nu}' = \partial_\mu A'_\nu - \partial_\nu A'_\mu$$

$$= \partial_\mu (A_\nu - \xi^\sigma (\partial_\sigma A_\nu)) - \partial_\nu (A_\mu - \xi^\sigma (\partial_\sigma A_\mu))$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - \xi^\sigma (\partial_\sigma A_\nu) - \xi^\sigma (\partial_\sigma A_\mu)$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - \xi^\sigma (\partial_\sigma A_\nu) - \xi^\sigma (\partial_\sigma A_\mu)$$

$$- (\partial_\sigma A_\nu - \partial_\sigma A_\mu) + (\partial_\sigma \xi^\lambda) A_\lambda + (\partial_\sigma \xi^\lambda) A_\lambda$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - (\partial_\sigma \xi^\lambda) A_\sigma + (\partial_\sigma \xi^\lambda) A_\sigma - \xi^\sigma \partial_\sigma A_\mu + \xi^\sigma \partial_\sigma A_\nu$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - (\partial_\sigma \xi^\lambda) A_\sigma + (\partial_\sigma \xi^\lambda) A_\sigma$$

$$= F_{J\nu} - \delta_\nu^\sigma (F_{\sigma J')} - \delta_\lambda^\lambda (F_{\lambda J'}) - \xi^\sigma \partial_\sigma (A_\mu - \partial_\nu A_\mu).$$

Finally, if we change some of the dummy indices and rearrange the terms, we get

$$F_{J\nu} \rightarrow F_{J\nu} - \xi^\sigma \partial_\sigma F_{J\nu} - (\partial_\lambda \xi^\mu) F_{J\lambda \nu} - (\partial_\nu \xi^\lambda) F_{J\mu \nu}.$$  (78)

As expected, $F_{J\nu}$ transforms as defined by the tensor definition for Lie derivatives, but we found this by directly performing the diffeomorphism on $A_\nu$. It is then safe to assume that $F_{J\nu}$ will take the form

$$F_{J\nu} \rightarrow F_{J\nu} - \xi^\sigma \partial_\sigma F_{J\nu} + (\partial_\lambda \xi^\mu) F_{J\lambda \nu} + (\partial_\nu \xi^\lambda) F_{J\mu \nu}.  (79)$$

So using our transformation definitions (78) and (79) we get

$$F_{J\nu} F^{\mu \nu} \rightarrow \left[ F_{J\nu} - \xi^\sigma \partial_\sigma F_{J\nu} - (\partial_\lambda \xi^\mu) F_{J\lambda \nu} - (\partial_\nu \xi^\lambda) F_{J\mu \nu} \right]$$

$$\left[ F^{\mu \nu} - \xi^\sigma \partial_\sigma F^{\mu \nu} + (\partial_\lambda \xi^\mu) F^{\lambda \nu} + (\partial_\nu \xi^\lambda) F^{\mu \lambda} \right]$$

$$\rightarrow \left[ F_{J\nu} F^{\mu \nu} - F_{J\nu} \xi^\sigma (\partial_\sigma F^{\mu \nu}) + F_{\nu J} (\partial_\lambda \xi^\mu) F^{\lambda \nu} + F_{\nu J} (\partial_\nu \xi^\lambda) F^{\mu \lambda} \right]$$

$$+ \left[ -\xi^\sigma (\partial_\sigma F_{J\nu}) F^{\mu \nu} + 0 + 0 + 0 \right] - \left[ -\xi^\sigma (\partial_\nu F_{J\nu}) F^{\mu \nu} + 0 + 0 + 0 \right]$$

$$+ \left[ -\xi^\sigma (\partial_\lambda F_{J\nu}) F^{\mu \nu} + 0 + 0 + 0 \right].$$

where the 0's are due to $(\xi^\mu)^2$ and higher terms being thrown away. Grouping the terms up again, we get

$$\rightarrow \left[ F_{J\nu} F^{\mu \nu} - 2\xi^\sigma (\partial_\sigma F_{J\nu}) F^{\mu \nu} + (\partial_\lambda \xi^\mu) F_{\nu J} F^{\lambda \nu} + (\partial_\nu \xi^\lambda) F_{\nu J} F^{\mu \lambda} \right]$$

$$- (\partial_\lambda \xi^\mu) F_{\nu J} F^{\mu \nu} - (\partial_\nu \xi^\lambda) F_{\nu J} F^{\mu \nu}]$$

$$\rightarrow \left[ F_{J\nu} F^{\mu \nu} - \xi^\sigma \partial_\sigma (F_{J\nu} F^{\mu \nu}) + 2(\partial_\lambda \xi^\mu) F_{\nu J} F^{\lambda \nu} - 2(\partial_\nu \xi^\lambda) F_{\nu J} F^{\mu \lambda} \right]$$

$$\rightarrow \left[ F_{J\nu} F^{\mu \nu} - \xi^\sigma \partial_\sigma (F_{J\nu} F^{\mu \nu}) \right].$$

Finally, we see

$$\frac{1}{4} F_{J\nu} F^{\mu \nu} \rightarrow \frac{1}{4} F_{J\nu} F^{\mu \nu} - \frac{1}{4} \xi^\sigma \partial_\sigma (F_{J\nu} F^{\mu \nu}).$$  (82)

$F_{J\nu}$ transforms properly under diffeomorphism, and thus the Lagrangian in (74) transforms properly under diffeomorphism and we have diffeomorphism invariance! In the next section we will examine the consequences of breaking such symmetries through the mechanism of spontaneous symmetry breaking.
6 Spontaneous Symmetry Breaking (SSB)

In this section, we will give some background in SSB in particle physics. In section (7), we will examine a model in which the diffeomorphism symmetry is spontaneously broken. Spontaneous Symmetry Breaking is a mechanism that is a fundamental feature in the Standard Model of particle physics. There are a number of phenomenological conservation laws, which reflect the prevalence of exact symmetries in nature. In the context of Lagrangian field theory, exact symmetry is characterized by the following two conditions: First, the Lagrangian must be invariant under the symmetry in question, and secondly, unique physical vacuums are invariant under the symmetry transformations. Spontaneous Symmetry Breaking (SSB) is a mechanism which leaves symmetry in the dynamic system, i.e. the Lagrangian, but not at unique physical vacuum values. This is achieved by spontaneously picking a unique physical vacuum out of a set of possible solutions, which essentially breaks the symmetry of the theory.

We encounter concepts of symmetry breaking in our daily lives. Think of a round table with plates, silverware, and all the glasses set equally between the plates. Once everyone sits down at the table, you can either choose to drink from the glass on your right, or you can choose to drink from the glass on your left. There is symmetry in the system, such that you can choose to drink from either glass. However, the instant you choose to drink from the glass on the right, everyone else in the circle must necessarily drink from the glass on their right as well and the symmetry is spontaneously broken. The act of picking a particular solution from the symmetrical set of possible solutions is an example of spontaneous symmetry breaking.

Mathematically, we can consider certain transformations that leave a system unchanged. These are said to be symmetries of theory and we will note that the symmetry in the Lagrangian holds, but not in the ground state solution, often called the vacuum.

In the Standard Model of particle physics, SSB has well-known consequences. In the electroweak model, the gauge symmetry SU(2) x U(1) is spontaneously broken. Here, we will consider SSB of an SO(2) (or U(1)) gauge theory as a simplified example. We can view the results from global and local transformations of the SO(2) gauge theory, which we will discuss in more detail in sections (6.2) and (6.3). A global transformation is a transformation that is the same throughout all the space of the manifold, whereas, a local transformation is a transformation that is defined uniquely at every point in the manifold. The distinction between global and local transformations is important because each case gives us a different result in the case of the SO(2) gauge theory.

In this section, we will examine the Goldstone Theorem, which states that when a global continuous symmetry is spontaneously broken, the theory must have massless modes, which are typically called Nambu-Goldstone modes. For a local spontaneously broken symmetry, we will see that the Nambu-Goldstone modes get “eaten” and massless gauge fields acquire mass. This process is called the Higgs Mechanism. We will start off by examining SSB in the simplest case for a discrete symmetry. Then we will extend the concept of SSB to continuous symmetries, where we examine the results of SSB in the global case for the SO(2) gauge theory, followed by the results of SSB in the local case of SO(2) gauge theory.
6.1 SSB for Discrete Symmetry

In this section, we examine SSB for the discrete symmetry of parity. Consider the Lagrangian

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - V(\phi), \]  

(83)

where \( V(\phi) \) is defined such that

\[ V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4 \]  

(84)

for \( \lambda > 0 \). We see that (83) is unchanged by a parity transformation

\[ \phi \rightarrow -\phi. \]  

(85)

Eq. (85) is, therefore, a symmetry of this theory and since parity is discrete, we call (85) a discrete symmetry of the theory.

If \( m^2 > 0 \), then \( V \) has a unique minimum at \( \phi = 0 \), which we could write as

\[ \langle \phi \rangle = 0. \]  

(86)

The quantity \( \langle \phi \rangle \) is the vacuum expectation value, abbreviated as vev, and represents the ground state of the system. Let's consider small oscillations, \( \epsilon \), about the vacuum

\[ \phi = \langle \phi \rangle + \epsilon, \]  

(87)

where \( \langle \phi \rangle = 0 \) as defined above. It is the small oscillations, \( \epsilon \), that describe the mode of the theory. By substituting (87) into (83) and keeping terms to 2nd order in \( \epsilon \), we get

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \epsilon)(\partial^\mu \epsilon) - \frac{1}{2} m^2 \epsilon^2 + O(\epsilon^3). \]  

(88)

From our analysis of the scalar case of classical field theory, we recognize that this Lagrangian describes \( \epsilon \) as a free massive particle with mass equal to \( m \). All of this seems familiar to processes we have done before.

If we want to induce spontaneous symmetry breaking, we must look at the case when \( m^2 < 0 \). We can note that this doesn't change the symmetry we get from (85), since we are only changing requirements on \( m \). However, now with \( m^2 < 0 \), the potential \( V \) has two possible minimums (or vev's),

\[ \langle \phi \rangle = \pm \sqrt{\frac{-m^2}{\lambda}}. \]  

(89)

We see that there is currently no preference to pick one minimum over the other, and thus we spontaneously pick one. Let's pick

\[ \langle \phi \rangle = \sqrt{\frac{-m^2}{\lambda}} = v. \]  

(90)

If we redefine our coordinates with respect to the new vacuum we chose, we get

\[ \phi = \langle \phi \rangle + \phi', \]
\[ \phi' = \phi - \langle \phi \rangle \]
\[ \phi' = \phi - v. \]  

(91)
This definition is useful because we can see that the vacuum for $\phi'$ is

$$\langle \phi' \rangle = 0.$$  

(92)

Therefore, the Lagrangian of (83) in terms of $\phi'$ is

$$L = \frac{1}{2} (\partial_{\mu} \phi')(\partial^{\mu} \phi') - (-m^2)(\frac{\phi'^4}{4} + \frac{\phi'^3}{3} + \phi'^2 - \frac{v^2}{4}).$$  

(93)

Notice now how the transformation (85) does not leave this Lagrangian the same because of the $\phi'^3$ term. This step illustrates SSB, in that, once we picked a specific vev, the symmetry becomes hidden! Furthermore, if we continue by considering small oscillations, $\epsilon$, about the vacuum

$$\phi' = \epsilon$$  

(94)

and plug this into (93) we end up with

$$L = \frac{1}{2} (\partial_{\mu} \epsilon)(\partial^{\mu} \epsilon) - \frac{1}{2} (-2m^2)\epsilon^2 + O(\epsilon^2).$$  

(95)

We can recognize that this provides us with a free massive particle with its mass^2 equal to $-2m^2$. Recall, for this step we assumed $m^2 < 0$, and thus our particle actually has positive mass, and is therefore real.

This entire scenario was for the discrete parity symmetry in (85). Many methods we used in this section, such as spontaneously breaking symmetry by choosing a specific vacuum out of a set of solutions, extend into the case for continuous symmetries. However, one element unique to the global continuous case is the application of Goldstone's theorem, as mentioned earlier. Recall that Goldstone's theorem states that whenever a global continuous symmetry is spontaneously broken, the theory will have massless modes. We can verify this theorem by looking at the global SO(2) gauge theory.

6.2 Global SO(2) Gauge Theory

In this section, we will examine the results for spontaneous symmetry breaking of the global SO(2) gauge theory. In breaking a continuous global symmetry, we expect to find massless modes by
Goldstone's theorem. In the SO(2) theory, we indeed verify the existence of these massless modes, referred to as Nambu-Goldstone modes, but we also find massive modes, referred to as Higgs modes. In this section, we will compare these resulting modes and examine what they do in the global SO(2) gauge theory. We start our examination with discussing the classification of SO(2) gauge theory.

For an N-dimensional real-valued vector \( X = (X_1, X_2, ..., X_N) \), a gauge transformation that maintains
\[
X^2 = \bar{X} \cdot \bar{X}
\]
(96)
is called an O(N) transformation, where the O is for orthogonal. If in addition,
\[
det(O) = 1
\]
(97)
then the gauge transformation is called an SO(N) transformation, where the S is for special. The SO(2) gauge transformation is then a special orthogonal gauge transformation of dimension 2.

The Lagrangian used in the SO(2) gauge theory is
\[
\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi) \cdot (\partial^{\mu} \phi) - V(\phi \cdot \phi),
\]
(98)
where \( \phi \) is composed of 2 scalar fields
\[
\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}
\]
(99)
The Lagrangian of (98) is invariant under global continuous SO(2)
\[
\phi \rightarrow \phi' = R \phi,
\]
(100)
where
\[
R = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}
\]
(101)
We define \( V(\phi \cdot \phi) \) as we did in the discrete symmetry case before,
\[
V(\phi \cdot \phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4,
\]
(102)
but here in the SO(2) gauge theory, we want to note that \( \phi^2 \) really means
\[
\phi^2 = \phi \cdot \phi = (\phi_1)^2 + (\phi_2)^2.
\]
(103)

As we saw in the previous example, if we look at when \( m^2 > 0 \), we have a unique minimum and no SSB can occur. Thus, we choose to induce SSB by looking at the case when \( m^2 < 0 \). This gives us
\[
\langle \phi \rangle^2 = \frac{-m^2}{\lambda} = v^2
\]
\[
\langle \phi_1 \rangle^2 + \langle \phi_2 \rangle^2 = v^2.
\]
(104)
We see that the equation implies that we have a "ring" of possible minima to chose from for our vev! (see figure on next page)
We can spontaneously break the symmetry by choosing the minimum to be defined at
\[ \langle \phi \rangle = \begin{pmatrix} u \\ 0 \end{pmatrix}. \] (105)

Let's redefine our coordinates with respect to the new vacuum we just chose to get
\[ \phi' = \phi - \langle \phi \rangle = \begin{pmatrix} \phi^1 - u \\ \phi^2 \end{pmatrix}. \] (106)

Similar to the discrete symmetry case, we conveniently get that the vev for $\phi'$ is simply
\[ \langle \phi' \rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \] (107)

Unlike the discrete symmetry case, SO(2) is 2 dimensional and thus, to consider small oscillations for this theory, we have to consider oscillations in both the $\phi^1$ and $\phi^2$ direction. We will define small oscillations in the $\phi^1$ direction as $\eta$ and define small oscillations in the $\phi^2$ direction as $\xi$. Thus, when we consider small oscillations, $\langle \eta \rangle$, about the vacuum and we get
\[ \phi' = \langle \phi' \rangle + \begin{pmatrix} \eta \\ \xi \end{pmatrix}. \] (108)

We want to express our Lagrangian in terms of these oscillations. We start with
\[ \partial_\mu \phi = \partial_\mu \phi' = \begin{pmatrix} \partial_\mu \eta \\ \partial_\mu \xi \end{pmatrix}, \] (109)

and
\[ \phi = \begin{pmatrix} u + \eta \\ \xi \end{pmatrix}, \]
\[ \phi^2 = (u + \eta)^2 + \xi^2. \] (110)
Plugging (110) into the potential defined in (102) gives

$$V(\phi \cdot \phi) = \frac{1}{2} m^2 [(\nu + \eta)^2 + \xi^2] + \frac{1}{4} \lambda[(\nu + \eta)^2 + \xi^2]^2$$

$$= \frac{1}{2} m^2 [\nu^2 + 2\nu\eta + \eta^2 + \xi^2] + \frac{1}{4} \lambda [\nu^2 + 2\nu\eta + \eta^2 + \xi^2]^2$$

$$= (m^2\nu + \lambda\nu^2)\eta + \left(\frac{1}{2} m^2 + \frac{3}{2} \lambda\nu^2\right)\eta^2 + \left(\frac{1}{2} m^2 + \frac{1}{2} \lambda\nu^2\right)\xi^2 + ...$$

where the ... accounts for $O(\xi^3), O(\eta^3)$ and higher terms. Recall that $\nu^2 = -\frac{m^2}{\lambda}$, so essentially

$$V(\phi \cdot \phi) = 0 - m^2\eta^2 + 0 + ...$$

Altogether, we end up with

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \eta)(\partial^\mu \eta) - \frac{1}{2} (-2m^2)\eta^2 + \frac{1}{2} (\partial_\mu \xi)(\partial^\mu \xi) + ...$$

From the previous cases we have studied, we can see that this Lagrangian includes a massive scalar, $\eta$, with mass $^2 = -2m^2$. This massive scalar is the Higgs mode and the oscillations take the Higgs mode up and down the sides of the potential well. The Lagrangian also contains an additional third term, which accounts for a massless mode, $\xi$. This massless mode is the Nambu-Goldstone mode and the oscillations keep the massless NG mode in the ring of potential minima.

Figure 7: Graphs of the motion for the NG and Higgs modes of the SO(2) gauge theory.

Finding a massless mode is precisely what Goldstone's theorem predicted would happen when we spontaneously broke the continuous SO(2) symmetry! This entire derivation was for the global SO(2) theory, and ignores the Higgs mechanism. To see the Higgs mechanism occur, we must look at the local SO(2) theory.

6.3 Local SO(2) Gauge Theory

In this section, we will examine the local SO(2) gauge theory and make note of any differences from the global case. Among these differences, we will find a massive gauge field, which has acquired its mass by "eating" the NG modes from the global case. This process by which the gauge fields acquire mass is known as the Higgs mechanism and occurs specifically here in the local case. Local gauge transformations are generalizations of global transformations, and the generalization
requires us to introduce gauge-covariant derivatives. We won’t discuss the mathematical details in depth, see Ref. [6], but we can examine why this is necessary by looking at the local SO(2) gauge theory.

For local SO(2) gauge theory

\[ \phi \rightarrow \phi' = R(x)\phi, \quad (114) \]

such that the 2x2 matrix \( R(x) \) is locally dependent:

\[ R = e^{i\alpha(x)}T, \quad (115) \]

where \( \alpha(x) \) is a locally dependent constant and where \( T \) is the 2x2 matrix

\[ T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \quad (116) \]

\( T \) is referred to as a generator of the theory (see ref. [6] for more on generators). To keep the theory invariant under this transformation, we need gauge-covariant derivatives, defined as

\[ D_\mu = \partial_\mu + igA_\mu. \quad (117) \]

Then we see that under the gauge transformation from (114), we would get

\[ D_\mu \phi \rightarrow D'_\mu \phi' = RD_\mu \phi. \quad (118) \]

\[ A_\mu \rightarrow A'_\mu = A_\mu + \frac{i}{g}(\partial_\mu R)R^{-1} = A_\mu - \frac{1}{g}(\partial_\mu \alpha). \quad (119) \]

By replacing \( \partial_\mu \) in the SO(2) Lagrangian with our covariant derivatives, our Lagrangian becomes

\[ \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu \phi \cdot D^\mu \phi - V(\phi \cdot \phi), \quad (120) \]

where once again we use

\[ V(\phi \cdot \phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4. \quad (121) \]

The Lagrangian of (120) is invariant under the local gauge transformation we defined in (114). To induce the SSB, we once again look at the case when \( m^2 < 0 \). We get the vacuum expectation value

\[ \langle \phi \rangle^2 = -\frac{m^2}{\lambda} = v^2. \quad (122) \]

We spontaneously break SO(2) symmetry by choosing

\[ \langle \phi \rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}. \quad (123) \]

Now let’s find small excitations \( \xi \) and \( \epsilon \) for the theory, where \( \xi \) is an excitation in the \( \phi^1 \) direction and \( \epsilon \) is an excitation in the \( \phi^2 \) direction. We can start this process by noting that since

\[ e^{i\alpha T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\alpha \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + \ldots, \]

(124)
Since $\alpha$ is small, the $O(\alpha^2)$ terms and higher will vanish, and we'd be left with,

$$e^{i\alpha T} = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}.$$  \hfill (125)

We can use the SO(2) symmetry to write an arbitrary field $\phi$ as

$$\phi = R^{-1}\phi',$$  \hfill (126)

where

$$\phi' = \begin{pmatrix} 0 \\ u + \epsilon \end{pmatrix}.$$  \hfill (127)

Now, if we let $\alpha = \frac{\epsilon}{\nu}$, we would get

$$R^{-1} = e^{i\alpha T} \approx \begin{pmatrix} 1 & -\frac{\epsilon}{\nu} \\ \frac{\epsilon}{\nu} & 1 \end{pmatrix}.$$  \hfill (128)

Substituting (128) into (126), we get

$$\phi = R^{-1}\phi' = \begin{pmatrix} 0 \\ -\frac{\epsilon}{\nu} \end{pmatrix} \begin{pmatrix} 0 \\ u + \epsilon \end{pmatrix} = \begin{pmatrix} -\epsilon \\ (u + \epsilon) \end{pmatrix}.$$  \hfill (129)

Eq. (129) describes the excitations $\xi$ and $\epsilon$ about the vev in (123)!

Similar to the steps we took in the global SO(2) gauge theory, we can work our excitations $\xi$ and $\epsilon$ back into the Lagrangian. Substituting in (129) gives

$$\mathcal{L} = \frac{1}{2}D_\mu (R^{-1}\phi')D^\mu (R^{-1}\phi') - V((R^{-1}\phi')^2) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \hfill (130)$$

The Lagrangian in (130) is gauge invariant, and thus we can perform our gauge transformations:

$$\phi \rightarrow R\phi = R(R^{-1}\phi') = \phi',$$

$$A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{g}(\partial_\mu \alpha). \hfill (131)$$

Substituting these transformations into (130) would give

$$\mathcal{L} = \frac{1}{2}(D'_\mu \phi')(D'^\mu \phi') - V(\phi'^2) - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu}. \hfill (132)$$

Let's look at each piece of this Lagrangian, starting with the potential.

$$V(\phi'^2) = V((u + \epsilon)^2)$$

$$= \frac{1}{2}m^2(u + \epsilon)^2 + \frac{1}{4}\lambda(u + \epsilon)^4$$

$$= \frac{1}{2}m^2(u^2 + 2u\epsilon + \epsilon^2) + \frac{1}{4}\lambda(u^4 + 4u^3\epsilon + 6u^2\epsilon^2 + ...)$$

$$= \epsilon(m^2u + \lambda u^3) + \epsilon^2(\frac{1}{2}m^2 + \frac{3}{2}\lambda u^2) + ... \hfill (133)$$
Note that, \( v^2 = -\frac{m^2}{2} \), and thus
\[
V(\phi^2) = \epsilon^2 \left( \frac{1}{2} m^2 - \frac{3}{2} m^2 \right)
= -m^2 \epsilon^2
\]
\[
= \frac{1}{2} (-2m^2) \epsilon^2.
\] (134)

Now look at the gauge-covariant derivatives.
\[
D'_\mu \phi' = (\partial_\mu + igA'_\mu) \phi'
= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_\mu + \begin{pmatrix} 0 & -gA'_\mu \\ gA'_\mu & 0 \end{pmatrix} \begin{pmatrix} 0 \\ (v + \epsilon) \end{pmatrix}
= \begin{pmatrix} 0 \\ (\partial_\mu \epsilon) + (-gA'_\mu(v + \epsilon)) \end{pmatrix}
= \begin{pmatrix} 0 \\ (-gA'_\mu(v + \epsilon)) \end{pmatrix}.
\] (135)

Thus, summing out the gauge-covariant derivatives would give
\[
D_\mu \phi' \cdot D^{\mu'} \phi' = (-gA'_\mu(v + \epsilon)) \left( \begin{pmatrix} 0 \\ (\partial_\mu \epsilon) \end{pmatrix} \right) \begin{pmatrix} -gA'_\mu(v + \epsilon) \\ \partial_\mu \epsilon \end{pmatrix}
= g^2 (A'_\mu A'^\mu) + (\partial_\mu \epsilon)(\partial^{\mu} \epsilon)
\] (136)

Finally, putting (134) and (136) back into our Lagrangian from (132) would give the result
\[
\mathcal{L} = \frac{1}{2} \partial_\mu e \partial^{\mu} e - \frac{1}{2} (-2m^2) \epsilon^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} g^2 v^2 A'_\mu A'^\mu + ..., \] (137)

where the ... is 3rd order and higher interactions.

From the previous cases that we have studied, we notice the first two terms of the Lagrangian attribute a massive scalar, \( \epsilon \), with mass\(^2 = (-2m^2) \). This massive scalar is the Higgs mode, which we saw in the global case. However, unlike in the global case, we have no NG modes in the local SO(2) theory! Instead, we have some additional terms that did not show up in the global case. The third and fourth term of the Lagrangian attribute a massive gauge field, \( A'_\mu \). Therefore, performing a local SO(2) transformation shows that the NG mode of the global case gets swapped for a massive gauge field \( A'_\mu \). This process by which massless fields acquire mass is known as the Higgs mechanism and is a key result of moving from the global SO(2) to local SO(2)! Note that the Higgs mechanism requires the massless NG mode, since the gauge field cannot acquire mass unless it “eats” up the would-be NG mode from the global case. Now that we have seen how global and local SO(2) gauge theories work in comparison, we can use the same techniques here to examine another more complicated model, called the Bumblebee Model.

7 SSB for Bumblebee Model

In this section of the thesis, we introduce and examine the specific model we worked with. This model is known as the Bumblebee Model and was originally theorized as one of the simplest cases.
of SSB emerging from String Theory, see Ref. [4]. The theory has a special hybrid Lagrangian that considers gravity and a vector field, along with spontaneous diffeomorphism symmetry breaking. We will explore what happens to the BB model under spontaneous diffeomorphism breaking and see if there are analogues between this model and the Standard Model of particle physics.

7.1 Bumblebee Model (Classical Field Theory)

The Lagrangian for the Bumblebee model is

\[ \mathcal{L} = \frac{1}{16\pi G} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(A_\mu A^\mu - a^2). \]  

(138)

Notice how this Lagrangian is a combination of the GR free space Lagrangian and the kinetic term for a massless vector field that we encountered earlier in the EM case. For this model, the potential is a function of \( A_\mu A^\mu \) and defined such that

\[ V(A_\mu A^\mu - a^2) = \frac{1}{2} \kappa (A_\mu A^\mu - a^2)^2, \]  

(139)

where \( \kappa \) is some constant. Note that the form of this potential destroys local U(1) gauge invariance and hence this is not Maxwell’s theory, despite the fact that the vector piece resembles that of the EM case.

Let’s vary the Lagrangian with respect to \( g^{\mu\nu} \) and \( A_\mu \) to see the theory’s equations of motion before introducing a vacuum expectation value. We know how the first two terms of (138) turn out when we vary with respect to \( g^{\mu\nu} \) and \( A_\mu \), but we still need to examine what happens to the potential under these variations. We start off by noticing that the potential is actually a function of both \( A_\mu \) and \( g^{\mu\nu} \) because it can be written as

\[ V = \frac{1}{2} \kappa (A_\rho g^{\rho\sigma} A_\sigma - a^2)^2. \]  

(140)

If we vary (140) with respect to \( A_\mu \), we get

\[ \frac{\delta V}{\delta A_\mu} = \kappa (A_\rho g^{\rho\sigma} A_\sigma) \frac{\delta}{\delta A_\mu} (A_\mu g^{\mu\nu} A_\nu - a^2) \]

\[ = \kappa (A_\rho g^{\rho\sigma} A_\sigma) 2A_\mu \]

\[ = 2\kappa A^\mu (A_\sigma A_\sigma - a^2). \]  

(141)

Likewise, if we vary (140) with respect to \( g^{\mu\nu} \), we get

\[ \frac{\delta V}{\delta g^{\mu\nu}} = \kappa (A_\rho g^{\rho\sigma} A_\sigma) \frac{\delta}{\delta g^{\mu\nu}} (A_\mu g^{\mu\nu} A_\nu - a^2) \]

\[ = \kappa A_\mu A_\nu (A_\rho g^{\rho\sigma} A_\sigma). \]  

(142)

Thus, extending what we know from the equations of motion in the EM case, we get

\[ \Box A^\mu - \partial^\nu \partial^\rho A_\rho - 2A_\mu (A_\sigma A_\sigma - a^2) = 0. \]  

(143)
Similarly, extending what we know from the equations of motion in the GR free space case, we get

$$G_{\mu\nu} = 8\pi G [F_{\mu\lambda} F_{\nu\lambda} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - 2\kappa A_\mu A_\nu (A^\alpha - a^2)].$$  \(144\)

These define the equations of motion for $A_\mu$ and $g_{\mu\nu}$ in the BB model, before looking at any effects from spontaneous symmetry breaking, which we will examine in the next section.

### 7.2 SSB in the Bumblebee Model

In this section, we examine the effects of spontaneously breaking the diffeomorphism symmetry of the theory. We hope to find results that we can compare with the Standard Model of particle physics and examine the implications of the comparisons in greater detail. We can start by noticing that the potential in (139) has a minimum when

$$A_\mu A^\mu - a^2 = 0.$$  \(145\)

Unlike the SO(2) theory, we do not have $\langle A_\mu \rangle = 0$. Instead we want $\langle A_\mu \rangle$ to be equal to a constant vector. Let's call

$$\langle A_\mu \rangle = a_\mu,$$  \(146\)

such that

$$a_\mu a^\mu = a^2.$$  \(147\)

Eq. (147) requires that $a_\mu$ must be a timelike vector. Let's spontaneously pick $a_\mu$ such that

$$a_\mu = (a, 0, 0, 0).$$  \(148\)

Defining $a_\mu$ as the constant above spontaneously breaks the diffeomorphism of the theory since we cannot perform a diffeomorphism on a constant!

With the symmetry broken, let's look at the excitations about the vacuum to examine the question of whether there are NG and Higgs modes and what their fates are. Let's define the excitation for $A_\mu$ as

$$A_\mu = a_\mu + \epsilon_\mu,$$  \(149\)

where $\epsilon_\mu$ is an excitation about the vacuum. Recall that the potential is a function of both $A_\mu$ and $g^{\mu\nu}$. $V = V(A_\mu, g^{\mu\nu})$, and therefore we should also consider small excitations for $g^{\mu\nu}$ and $g_{\mu\nu}$. For $g^{\mu\nu}$ we have

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu},$$  \(150\)

where $\eta^{\mu\nu}$ is the expectation value ($g^{\mu\nu}$) and $h^{\mu\nu}$ is a small oscillation about the vacuum. For the covariant version, we have

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$  \(151\)

where $\eta_{\mu\nu}$ is the vev ($g_{\mu\nu}$) and $h_{\mu\nu}$ is a small oscillation about the vacuum. Therefore, when we talk about the vacuum, we really have 2 vev's

$$\langle A_\mu \rangle = a_\mu,$$  
$$\langle g^{\mu\nu} \rangle = \eta^{\mu\nu}.$$  \(152\)
where the relation from (147) still holds, but in terms of $a_\mu$ and $\eta^{\mu\nu}$ is

$$a_\mu \eta^{\mu\nu} a_\nu = a^2.$$

(153)

In order to examine the modes that describe the theory, we want to get the Lagrangian in terms of the small oscillations $\epsilon_\mu$ and $h_{\mu\nu}$ to quadratic order. First, let’s substitute (149) and (151) into the potential function, which gives

$$V = \frac{1}{2} \kappa [A_\mu g^{\mu\nu} A_\nu - a^2]^2$$

(154)

$$= \frac{1}{2} \kappa [(a_\mu + \epsilon_\mu)(\eta^{\mu\nu} - h^{\mu\nu})(a_\nu + \epsilon_\nu) - a^2]^2.$$  

We will make life easier by noting the terms in the potential are squared. The restriction for the Lagrangian to be of quadratic order then restricts the terms inside the square of the potential to be linear. Imposing this restriction leaves us with

$$V = \frac{1}{2} \kappa [a^{\mu\nu} a_\nu + 2a^\mu \epsilon_\mu - a_\mu h^{\mu\nu} a_\nu - a^2]^2$$

(155)

$$= \frac{1}{2} \kappa [2a^{\mu\nu} (\epsilon_\mu - \frac{1}{2} h_{\mu\nu} a_\nu)]^2.$$

We saw earlier that the action is unchanged under an infinitesimal diffeomorphism. Let’s look at how $A_\mu$, $g_{\mu\nu}$, and $g^{\mu\nu}$ transform under diffeomorphism by substituting in our expressions from (149), (150) and (151). Note that the excitations that describe the Nambu-Goldstone modes are not technically diffeomorphisms, as we’ll see that they are reparametrizations that only appear to have the same form as diffeomorphisms, and therefore, will be called virtual diffeomorphisms. The NG modes are the field variations that stay within the minimum of the potential $V$. These can be found as the virtual diffeomorphisms that leave $V = V' = 0$ unchanged.

Recall the diffeomorphism transformation for $A_\mu$ was

$$A_\mu \rightarrow A_\mu - (\partial_\mu \xi^\alpha) A_\alpha - \xi^\alpha \partial_\alpha A_\mu,$$

(156)

with $\xi^\mu$ small. If we substitute $A_\mu = a_\mu + \epsilon_\mu$, then

$$a_\mu + \epsilon_\mu \rightarrow a_\mu + \epsilon_\mu - (\partial_\mu \xi^\alpha)(a_\alpha + \epsilon_\alpha) - \xi^\alpha \partial_\alpha (a_\mu + \epsilon_\mu)$$

(157)

$$\rightarrow a_\mu + \epsilon_\mu - (\partial_\mu \xi^\alpha) a_\alpha,$$

and we can conclude that

$$a_\mu \rightarrow a_\mu,$$

$$\epsilon_\mu \rightarrow \epsilon_\mu - (\partial_\mu \xi^\alpha) a_\alpha.$$

(158)

Likewise, for $g_{\mu\nu}$

$$g_{\mu\nu} \rightarrow g_{\mu\nu} - (\partial_\mu \xi^\alpha) g_{\alpha\nu} - (\partial_\nu \xi^\alpha) g_{\mu\alpha} - \xi^\alpha \partial_\alpha g_{\mu\nu}$$

$$\eta_{\mu\nu} + h_{\mu\nu} \rightarrow \eta_{\mu\nu} + h_{\mu\nu} - (\partial_\mu \xi^\alpha)(\eta_{\alpha\nu} + h_{\mu\nu})$$

(159)

$$- (\partial_\nu \xi^\alpha)(\eta_{\mu\alpha} + h_{\mu\alpha}) - \xi^\alpha \partial_\alpha (\eta_{\mu\nu} + h_{\mu\nu}).$$
and we can conclude that

\[
\begin{align*}
\eta_{\mu\nu} & \to \eta_{\mu\nu}, \\
h_{\mu\nu} & \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu.
\end{align*}
\] (160)

Analogously, for \( g^{\mu\nu} \) we just have the contravariant version

\[
\begin{align*}
\eta^{\mu\nu} & \to \eta^{\mu\nu}, \\
h^{\mu\nu} & \to h^{\mu\nu} - \partial^{\mu} \xi^{\nu} - \partial^{\nu} \xi^{\mu}.
\end{align*}
\] (161)

The key idea behind these results is that if we call

\[
\begin{align*}
\langle \epsilon_\mu \rangle & = 0, \\
\langle h_{\mu\nu} \rangle & = 0, \\
\langle h^{\mu\nu} \rangle & = 0,
\end{align*}
\] (162)

then we can consider small oscillations about these vacuum values, governed by (158), (160) and (161). The oscillations would be

\[
\begin{align*}
\epsilon_\mu & = -(\partial_\mu \xi^\alpha) a_\alpha, \\
h_{\mu\nu} & = -\partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \\
h^{\mu\nu} & = -\partial^{\mu} \xi^{\nu} - \partial^{\nu} \xi^{\mu}.
\end{align*}
\] (163)

Note that these appear to have the form of a diffeomorphism, but are technically just reparametrisations of \( \epsilon_\mu \) and \( h_{\mu\nu} \) and thus are not actually performed on the entire action as true diffeomorphisms.

Let's see what happens when we substitute (163) into the potential. If we simplify the potential as

\[
\begin{align*}
V & = \frac{1}{2} \kappa X^2 \\
V' & = \kappa X \\
X & = a^\mu (\epsilon_\mu - \frac{1}{2} h_{\mu\nu} a^\nu),
\end{align*}
\] (164)

we see that using (163) gives

\[
\begin{align*}
X & = a^\mu (-(\partial_\mu \xi^\alpha) a_\alpha - \frac{1}{2} (\partial_\mu \xi_\nu - \partial_\nu \xi_\mu) a^\nu) \\
& = -a^\mu (\partial_\mu \xi_\nu) a^\nu + \frac{1}{2} a^\mu (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) a^\nu \\
& = -a^\mu (\partial_\mu \xi_\nu) a^\nu + a^\mu (\partial_\mu \xi_\nu) a^\nu \\
& = 0.
\end{align*}
\] (165)

If \( X = 0 \) then \( V = 0 \) and \( V' = 0 \). This means, the fluctuations from (163) describe excitations that remain in the minimum of the potential. From the global SO(2) theory, we know that these must be the Nambu-Goldstone mode excitations! Thus, if the excitations in (163) describe NG excitations, then the small \( \xi^\mu \) must describe the NG fields. To denote the significance of the \( \xi^\mu \) fields, we will
notationally promote them to $\Xi^\mu$ fields, where these are the NG fields of the spontaneously broken BB model. When we apply this change to our excitations, we get the new notational form

$$\begin{align*}
\epsilon_\mu &= - (\partial_\mu \Xi) a^\nu, \\
h_\mu &= - \partial_\mu \Xi^\nu - \partial_\nu \Xi_\mu, \\
h^{\mu\nu} &= - \delta^{\mu\nu} - \partial^{\mu} \Xi^\nu.
\end{align*}$$

(166)

Now we can look at the original Lagrangian for the Bumblebee model in (138) and write it in terms of the $\Xi^\mu$ fields to observe what the NG fields are doing. We must first get (138) in terms of $\epsilon_\mu$ and $h_{\mu\nu}$, so that we can use the NG excitations in (166). For starters, since we are working with NG excitations, we will remain in the minimum of the potential and therefore $V = 0$. Thus, we only really need to worry about the remaining two terms of the Lagrangian,

$$\mathcal{L} = \frac{1}{16\pi G} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (167)$$

Looking at $F_{\mu\nu}$, we find

$$\begin{align*}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
&= \partial_\mu (a_\nu + \epsilon_\nu) - \partial_\nu (a_\mu + \epsilon_\mu) \\
&= \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu \\
&= \partial_\mu [-(\partial_\nu \Xi^\alpha) a_\alpha] - \partial_\nu [-(\partial_\mu \Xi^\alpha) a_\alpha] \\
&= - (\partial_\mu \partial_\nu \Xi^\alpha) a_\alpha + (\partial_\nu \partial_\mu \Xi^\alpha) a_\alpha \\
&= 0.
\end{align*}$$

(168)

$\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ simply vanishes from the Lagrangian under NG excitations! That just leaves the second term of (167). We can skip a lot of brute work, see Ref. [2], and take for granted that the remaining Lagrangian in terms of $h_{\mu\nu}$ is

$$\mathcal{L} = \frac{1}{2} \left[ (\partial_\mu h^{\mu\nu})(\partial_\nu h) - (\partial_\mu h^{\nu\rho})(\partial_\rho h^{\mu\sigma}) + \frac{1}{2} \eta^{\mu\nu}(\partial_\mu h^{\rho\sigma})(\partial_\rho h^{\mu\sigma}) - \frac{1}{2} \eta^{\mu\nu}(\partial_\mu h)(\partial_\nu h) \right], \quad (169)$$

where $h$ is the contraction

$$h = \eta^{\mu\nu} h_{\mu\nu} = h^\mu_\mu, \quad (170)$$

which, under NG excitations, yields

$$h = \eta^{\mu\nu} (-\partial_\mu \Xi_\nu - \partial_\nu \Xi_\mu) = -2 \partial_\mu \Xi^\mu. \quad (171)$$

Now let's use the NG excitations from (166) on the Lagrangian of (169). Ignoring the $\frac{1}{2}$ factor out in front, and breaking down the Lagrangian to look at each of its terms, we see

1st term: $(\partial_\mu h^{\mu\nu})(\partial_\nu h)$

$$\begin{align*}
&= (\partial_\mu (-\partial_\mu \Xi^\nu - \partial^\nu \Xi^\mu))[\partial_\nu (-2 \partial_\rho \Xi^\sigma)] \\
&= -[\partial_\mu \partial_\nu \Xi^\nu + \partial_\mu \partial_\nu \Xi^\mu] - 2 \partial_\nu \partial_\mu \Xi^\sigma \\
&= 2 \partial_\nu \partial_\mu \Xi^\nu \partial_\sigma \Xi^\sigma + \partial_\nu \partial_\mu \Xi^\mu \partial_\sigma \Xi^\sigma,
\end{align*}$$

(172)
\[ 2^{nd} \text{term} : - (\partial_{\mu} h^{\alpha \beta})(\partial_{\rho} h^{\mu \nu} h_{\alpha \beta}) \]
\[ = - (\partial_{\mu}[- \partial^{\xi} \Xi - \partial^{\eta} \Xi])(\partial_{\rho} h^{\mu \nu}(- \partial_{\xi} \Xi - \partial_{\eta} \Xi)) \]
\[ = \eta^{\mu \nu}[\partial_{\mu} \partial^{\alpha} \Xi \partial_{\rho} \partial_{\alpha} \Xi + \partial_{\rho} \partial^{\alpha} \Xi \partial_{\mu} \partial_{\alpha} \Xi] \]
\[ + \partial_{\mu} \partial^{\xi} \Xi \partial_{\rho} \partial_{\xi} \Xi + \partial_{\rho} \partial^{\xi} \Xi \partial_{\mu} \partial_{\xi} \Xi] \]
\[ = - \eta^{\mu \nu}[\partial_{\mu} \partial^{\alpha} \Xi \partial_{\rho} \partial_{\alpha} \Xi + \partial_{\rho} \partial^{\alpha} \Xi \partial_{\mu} \partial_{\alpha} \Xi] \]
\[ + \partial_{\mu} \partial^{\xi} \Xi \partial_{\rho} \partial_{\xi} \Xi + \partial_{\rho} \partial^{\xi} \Xi \partial_{\mu} \partial_{\xi} \Xi] \]
\[ = - \eta^{\mu \nu}[2 \partial_{\mu} \partial^{\alpha} \Xi \partial_{\rho} \partial_{\alpha} \Xi + 2 \partial_{\rho} \partial^{\alpha} \Xi \partial_{\mu} \partial_{\alpha} \Xi] \]
\[ + 2 \partial_{\mu} \partial^{\xi} \Xi \partial_{\rho} \partial_{\xi} \Xi + 2 \partial_{\rho} \partial^{\xi} \Xi \partial_{\mu} \partial_{\xi} \Xi] \]  \hspace{1cm} (173)

\[ 3^{rd} \text{term} : \frac{1}{2} \eta^{\mu \nu}(\partial_{\mu} h^{\alpha \beta})(\partial_{\nu} h^{\alpha \beta}) \]
\[ = \frac{1}{2} \eta^{\mu \nu}[- \partial_{\mu}(\partial^{\alpha} \Xi + \partial^{\rho} \Xi) \partial_{\nu}(\partial_{\alpha} \Xi + \partial_{\rho} \Xi)] \]
\[ = \frac{1}{2} \eta^{\mu \nu}[- \partial_{\mu} \partial^{\alpha} \Xi \partial_{\nu} \partial_{\alpha} \Xi + \partial_{\nu} \partial^{\alpha} \Xi \partial_{\mu} \partial_{\alpha} \Xi] \]
\[ = \frac{1}{2} \eta^{\mu \nu}[- \partial_{\mu} \partial^{\alpha} \Xi \partial_{\nu} \partial_{\alpha} \Xi + \partial_{\nu} \partial^{\alpha} \Xi \partial_{\mu} \partial_{\alpha} \Xi] \]
\[ + \partial_{\mu} \partial^{\xi} \Xi \partial_{\nu} \partial_{\xi} \Xi + \partial_{\nu} \partial^{\xi} \Xi \partial_{\mu} \partial_{\xi} \Xi] \]  \hspace{1cm} (174)

\[ 4^{th} \text{term} : - \frac{1}{2} \eta^{\mu \nu}(\partial_{\mu} h)(\partial_{\nu} h) \]
\[ = - \frac{1}{2} \eta^{\mu \nu}[- \partial_{\mu}(-2 \partial_{\xi} \Xi)] \]
\[ = - \frac{1}{2} \eta^{\mu \nu}[- \partial_{\mu}(-2 \partial_{\xi} \Xi)] \]  \hspace{1cm} (175)

Combining (173) with (174), we get several cancellations from dummy indices and end up with
\[ (173) + (174) = - \frac{1}{2} \eta^{\mu \nu}[4 \partial_{\mu} \partial^{\alpha} \Xi \partial_{\nu} \partial_{\alpha} \Xi] \]
\[ = - \frac{1}{2} \eta^{\mu \nu}[4 \partial_{\mu} \partial^{\alpha} \Xi \partial_{\nu} \partial_{\alpha} \Xi] \]  \hspace{1cm} (176)

Then adding (175) to the equation above gives
\[ (173) + (174) + (175) = - \frac{1}{2} \eta^{\mu \nu}[4 \partial_{\mu} \partial^{\alpha} \Xi \partial_{\nu} \partial_{\alpha} \Xi + 4 \partial_{\nu} \partial^{\alpha} \Xi \partial_{\mu} \partial_{\alpha} \Xi] \]
\[ = - 2[\partial^{\xi} \partial_{\xi} \Xi \partial_{\lambda} \partial_{\mu} \partial_{\lambda} \partial_{\nu} \Xi] \]  \hspace{1cm} (177)

Finally, adding in (172) to the equation gives
\[ (172) + (173) + (174) + (175) = 2[\partial_{\mu} \partial^{\alpha} \Xi \partial_{\nu} \partial_{\alpha} \Xi + \partial_{\nu} \partial^{\alpha} \Xi \partial_{\mu} \partial_{\alpha} \Xi] \]
\[ - \partial^{\xi} \partial_{\xi} \Xi \partial_{\nu} \partial_{\alpha} \Xi - \partial^{\xi} \partial_{\xi} \Xi \partial_{\mu} \partial_{\alpha} \Xi] \]
\[ = 2[\partial_{\mu} \partial^{\alpha} \Xi \partial_{\nu} \partial_{\alpha} \Xi - \partial^{\alpha} \partial_{\mu} \partial_{\alpha} \Xi \partial_{\nu} \partial_{\xi} \Xi] \]
\[ = 2[\partial_{\mu} \partial^{\alpha} \Xi \partial_{\nu} \partial_{\alpha} \Xi - \partial_{\mu} \partial^{\alpha} \Xi \partial_{\nu} \partial_{\alpha} \Xi] \]
\[ = 0. \]
The Lagrangian vanishes to zero under the NG excitations! Thus, we have witnessed the entire Lagrangian of the Bumblebee model vanishes under the Nambu-Goldstone excitations!

Furthermore, we can generalize to show that any covariant GR Lagrangian with spontaneous diffeomorphism breaking goes to zero for the NG mode excitations. Any covariant \( \mathcal{L} \) with spontaneous diffeomorphism breaking will consists of the components \( R_{\mu
u\rho\sigma} \) and the covariant derivative \( D_\mu A_\nu \); anything else is a combination or contraction of these, such as \( F_{\mu\nu} \). Therefore, if we can show that each of these components goes to zero under NG mode excitations, then we can show that the entire \( \mathcal{L} \) goes to zero in general. Let’s first look at

\[
R_{\mu
u\rho\sigma} = \eta_{\mu\lambda} \partial_\rho \Gamma^\lambda_{\nu\sigma} - \eta_{\nu\lambda} \partial_\sigma \Gamma^\lambda_{\mu\rho}.
\]  

(179)

In order to figure out \( R_{\mu
u\rho\sigma} \), we must first look at

\[
\Gamma^\rho_{\mu
u} = \frac{1}{2} \eta^{\rho\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})
= \frac{1}{2} \eta^{\rho\lambda} (\partial_\mu (\eta_{\nu\lambda} + h_{\nu\lambda}) + \partial_\nu (\eta_{\lambda\mu} + h_{\lambda\mu}) - \partial_\lambda (\eta_{\mu\nu} + h_{\mu\nu})] 
= \frac{1}{2} \eta^{\rho\lambda} (\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}).
\]

(180)

If we perform our excitation, \( h_{\mu\nu} = -\partial_\mu \Xi_\nu - \partial_\nu \Xi_\mu \), then we get

\[
\Gamma^\rho_{\mu
u} = \frac{1}{2} \eta^{\rho\lambda} (-\partial_\nu \Xi_\lambda - \partial_\lambda \Xi_\nu) + \partial_\nu (-\partial_\lambda \Xi_\mu - \partial_\mu \Xi_\lambda) - \partial_\lambda (-\partial_\mu \Xi_\nu - \partial_\nu \Xi_\mu)]
= \frac{1}{2} \eta^{\rho\lambda} [\partial_\nu \partial_\lambda \Xi_\mu - \partial_\lambda \partial_\nu \Xi_\mu - \partial_\nu \partial_\lambda \Xi_\mu + \partial_\lambda \partial_\nu \Xi_\mu + \partial_\nu \partial_\lambda \Xi_\mu]
= \frac{1}{2} \eta^{\rho\lambda} [-2 \partial_\nu \partial_\lambda \Xi_\mu].
\]

(181)

Therefore, we are left with the relation that

\[
\Gamma^\rho_{\mu
u} = -\partial_\nu \partial_\lambda \Xi_\mu.
\]

(182)

Note that it is not technically correct to talk about excitations in \( \Gamma^\rho_{\mu
u} \) to 1st order in \( \Xi \), since the Christoffel symbol is not technically a tensor. (182) is a purely mathematical relation we will use and contains no physical meaning by itself. We can substitute (182) into our definition for \( R_{\mu
u\rho\sigma} \)

\[
R_{\mu
u\rho\sigma} = \eta_{\mu\lambda} \partial_\rho (\partial_\nu \partial_\sigma \Xi_\lambda) - \eta_{\nu\lambda} \partial_\sigma (\partial_\mu \partial_\rho \Xi_\lambda)
= -\eta_{\mu\lambda} [\partial_\nu \partial_\sigma \partial_\rho \Xi_\lambda] + \eta_{\nu\lambda} [\partial_\mu \partial_\sigma \partial_\rho \Xi_\lambda].
\]

(183)

And we arrive at the conclusion

\[
R_{\mu
u\rho\sigma} = 0.
\]

(184)

As a side note, (184) has implications at the level of the equations of motion for the BB model as well. Recall that the equations of motion for the BB model include the Einstein tensor:

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R.
\]

(185)

\( R_{\mu\nu} \) and \( R \) are both contractions of \( R_{\mu
u\rho\sigma} \) and therefore (184) implies that the equations of motion for the BB model vanish under NG mode excitations as well!
We continue showing that any covariant GR Lagrangian vanishes to zero under NG excitations by looking at the covariant derivative (while throwing away terms with $\Xi \cdot \epsilon$)

\[
D_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda \\
= \partial_\mu (a_\nu + \epsilon_\nu) - \Gamma^\lambda_{\mu\nu} (a_\lambda + \epsilon_\lambda) \\
= \partial_\mu \epsilon_\nu + \partial_\mu \partial_\nu \Xi^\lambda (a_\lambda + \epsilon_\lambda) \\
= \partial_\mu (-\partial_\nu \Xi^\alpha) a_\alpha + \partial_\mu \partial_\nu \Xi^\lambda a_\lambda + \partial_\mu \partial_\nu \Xi^\lambda \epsilon_\lambda \\
= -\partial_\mu \partial_\nu \Xi^\alpha a_\alpha + \partial_\mu \partial_\nu \Xi^\lambda a_\lambda,
\]

and we arrive at the conclusion

\[
D_\mu A_\nu = 0. \tag{187}
\]

In summary, we not only concluded that the BB model Lagrangian vanishes to zero under NG mode excitations, but any covariant GR Lagrangian, constructed from contractions of $R_{\mu\nu\rho\sigma}$, $F_{\mu\nu}$, and $D_\mu A_\nu$ with each other and with $g_{\mu\nu}$, vanishes under NG mode excitations! In the following section, we will discuss the implications of this discovery.

8 Interpretations

What happens when a GR theory spontaneously breaks diffeomorphism invariance? We concluded the last section with the statement that any covariant Lagrangian in General Relativity must vanish to zero under Nambu-Goldstone mode excitations. In this section, we will discuss the implications of this statement, such as the absence of a Higgs mechanism, and rectify what appears to be a flaw in Goldstone’s theorem.

8.1 Goldstone’s Theorem revisited

Goldstone’s theorem states that when we break a continuous global symmetry, then we should find massless Nambu-Goldstone modes. If this theorem is true, then aren’t we guaranteed to find NG modes in our Bumblebee model? Since we do not find any NG modes in the BB model, does that mean Goldstone’s theorem is flawed? The answer is no.

As it turns out, there are a few subtle points that defend the integrity of Goldstone’s theorem. First off, Goldstone’s theorem is stated for global symmetries only. We had to necessarily introduce the concept of manifolds and diffeomorphisms to work with models in General Relativity. A diffeomorphism on a manifold is inherently local, and thus we technically have local diffeomorphism breaking. Since global is a subgroup of local, Goldstone’s theorem cannot guarantee that we will find NG modes under local symmetry breaking. Secondly, the proof for Goldstone’s theorem is done in flat spacetime and assumes Lorentz and diffeomorphism invariance. The BB model is in curved spacetime and breaks diffeomorphism invariance. These two points together provide reasonable explanation as to why we are not guaranteed NG modes through Goldstone’s theorem.
8.2 Absence of Higgs Mechanism

Recall that in the SO(2) theory the Higgs mechanism occurred when we moved from performing a global SO(2) transformation to a local one. The Higgs mechanism relied on the existence of a Nambu-Goldstone mode in the global transformation, since the gauge field in the local SO(2) transformation "eats" up the NG mode to acquire mass.

In our Bumblebee model, we showed that we could not possibly have an NG mode since the entire Lagrangian vanishes to zero under NG excitations. Without any NG modes, there cannot be a Higgs mechanism since the gauge fields in the local case have no would-be NG modes to "eat" up and acquire mass. Furthermore, we showed that any covariant GR Lagrangian vanishes under NG excitations and this result leaves no room for a Higgs mechanism for theories in General Relativity where diffeomorphisms are spontaneously broken!

Another argument for the absence of the Higgs mechanism arises if we go back and compare our BB model with the U(1) gauge theory. The Lagrangian for the U(1) gauge theory is

\[ L = \frac{1}{2} D_{\mu} \phi D^{\mu} \phi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - V(\phi \cdot \phi), \]  

where, as we have seen,

\[ V(\phi \cdot \phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4. \]

Note that we called this SO(2) theory before, but recall that SO(2) = U(1). The definition of the covariant derivatives here is

\[ D_{\mu} = \partial_{\mu} - iq A_{\mu}. \]

The mass term for \( A_{\mu} \) comes from the kinetic term we get when we square the covariant derivatives together

\[ D_{\phi} D^{\phi} \phi = ... \phi^2 A_{\mu} A^\mu. \]

The term \( A_{\mu} A^{\mu} \) indicates that the gauge boson is massive and we see the manifestation of the Higgs mechanism!

Now, we compare the U(1) gauge theory with our BB model. In the BB model, \( g_{\mu \nu} \) is our gauge field and \( A_\mu \) is the analogue to the scalar field \( \phi \) of the U(1) gauge theory. To have a Higgs mechanism in the BB model, the mass term for our gauge field \( g_{\mu \nu} \) would come from the covariant derivatives. The covariant derivatives in the BB model are defined as

\[ D_{\mu} = \partial_{\mu} - \Gamma_{\mu \nu}^\lambda, \]

where \( \Gamma_{\mu \nu}^\lambda \) is the analogue of \( iq A_{\mu} \) in the U(1) gauge theory. The mass term for \( g_{\mu \nu} \) would have to come from squaring the covariant derivatives together

\[ D_{\mu} A_{\nu} D^{\mu} A^\nu = ... \Gamma^2. \]

We get \( \Gamma^2 \) terms, but the big difference from here to the U(1) gauge theory is that the \( \Gamma^2 \) terms are not mass terms for \( g_{\mu \nu} \). This is because

\[ \Gamma_{\mu \nu} \propto g^{\lambda \sigma}(\partial_\mu g_{\sigma \nu} + \partial_\sigma g_{\mu \nu} - \partial_\nu g_{\mu \sigma}) \]

which clearly does not contribute any terms that would give mass to \( g_{\mu \nu} \). If the \( g_{\mu \nu} \) gauge field doesn't acquire mass, then this shows that there must not have been any conventional Higgs mechanism in the BB model!
8.3 Other Arguments

There have been various other papers written on the subject of spontaneous symmetry breaking and Goldstone's theorem. One paper examines Goldstone's theorem in relation to the effects of spontaneously breaking Poincaré symmetry in flat spacetime (see Ref [5]). The paper does indeed find massless NG modes, and therefore, a corresponding Higgs mechanism. However, we need not be alarmed with their findings since breaking Poincaré symmetry in flat spacetime, while counting as spontaneous symmetry breaking, does not affect diffeomorphism invariance.

In another paper, a group of physicists from Harvard assume the existence of a Nambu-Goldstone mode and then work backwards through a piecewise construction to create a diffeomorphism invariant theory (see Ref [1]). This may seem a little more alarming, as they have similar criteria, such as spontaneous diffeomorphism breaking, and yet they find an NG mode. The logical question that arises is, whose theory is correct? Although it is the first logical question to ask, it is slightly misleading because the two theories are constructed with a few subtle differences.

In the paper titled "Universal Dynamics of Spontaneous Lorentz Violation and a New Spin-Dependent Inverse-Square Law Force," the Harvard physicists end up with a Nambu-Goldstone mode, which they label as a scalar $\pi$. They also only consider spontaneous time diffeomorphism breaking and preserve spatial diffeomorphism invariance. As a result, their NG mode, $\pi$, has the requirement that it must transform as

$$\pi \rightarrow \pi - \xi_0,$$

but this is not how a proper scalar should transform! Also, their formulation leads to the construction of a Lagrangian that is not covariant. The implications of a GR Lagrangian that is not covariant would require developing some new theories in General Relativity altogether, and for our thesis, we have stuck to the assumption that GR Lagrangians should be covariant. With these reasons, we feel comfortable with the subtle discrepancies between the two theories to see them as two independently correct theories, rather than two opposing theories.

9 Summary and Conclusions

At the beginning of this thesis we set out to answer the following questions: Can GR be written as a gauge theory? If so, what is the relevant gauge transformation? We saw that we can indeed formulate GR as a gauge theory, where diffeomorphisms are the mathematical gauge transformations that leave the theory invariant. It is this property that leads many to refer to GR as a diffeomorphism invariant theory. After reviewing some fundamental results of the Standard Model of particle physics, we examined a specific gauge theory called the Bumblebee model, which emerges as one of the simplest cases of SSB from String Theory. The Bumblebee model features hybrid combinations of GR, a vector piece, and a potential with a degenerate set of minima. Spontaneously breaking the diffeomorphism symmetry of the BB model and examining the NG mode excitations showed that the BB Lagrangian vanished to zero. This implies that there is no Higgs mechanism in the BB model, and we extended this argument to show that any covariant GR theory lacks a Higgs mechanism. Further research can be done on examining theories of GR that do not have covariant Lagrangians, as well as taking a closer look at the Higgs mode of the BB model, both with the common goal of providing clues to developing a grand unifying model of physics which reconciles
the effects of gravity with quantum mechanics.

10 Glossary

10.1 Vectors, Tensors

Vector (Manifold) - Given some manifold \( M \), vectors are objects, defined at a specific point \( p \in M \), that operate on the space of all smooth functions at that point \( p \). The associated components of the vector, responsible for defining its direction, are determined by how the vector operates on the space of all smooth functions at that point \( p \). Mathematically, for a function \( f \) and vector \( V \)

\[
V : f \rightarrow \mathbb{R}^n.
\]  

(196)

Because a space - A collection of vectors that can be scaled and added.

Tangent space \( T_p \) - A vector space consisting of all the possible vectors at point \( p \), where \( p \in \text{manifold } M \). Elements of \( T_p \) can be referred to as contravariant vectors.

Dual Vector (1-form) - A dual vector is object that operates on vectors at a specific point in the manifold. Mathematically, for a dual vector \( W \) and a vector \( V \)

\[
W : V \rightarrow \mathbb{R}^n.
\]  

(197)

Cotangent space \( T^*_p \) - A dual space consisting of all possible dual vectors at a point \( p \). Elements of \( T^*_p \) can be referred to as covariant vectors.

Tensor - A multi-directional generalization of a vector that can contain covariant and contravariant vector components. Vectors and dual vectors are specific types of tensors of type \((0,1)\) and \((1,0)\) respectfully. Tensors transform properly under general coordinate transformations, that is

\[
T_{\mu_1...\mu_k}^{\nu_1...\nu_l} = \frac{\partial x^{\mu_1}}{\partial x^{\nu_1}} ... \frac{\partial x^{\mu_k}}{\partial x^{\nu_l}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1}} ... \frac{\partial x^{\nu_l}}{\partial x^{\nu_l}} T_{\mu_1...\mu_k}^{\nu_1...\nu_l}.
\]  

(198)

10.2 Classical Field Theory

Action - The integral of the Lagrangian density, or

\[
S = \int \mathcal{L} \, dx.
\]  

(199)

Principle of Least Action - States that the evolution of a system is such that small variations in the action should be equal to zero, or

\[
\delta S = \int \delta \mathcal{L} = 0.
\]  

(200)
**Auxiliary field** - When solving for equations of motion of a field theory, an auxiliary mode is a non-propagating mode and thus has no physical significance.

**Gauge Parameter** - A redundant degree of freedom in one of the field variables. The gauge parameter can be arbitrarily set to aid in solving for a specific theories modes. In the EM case, the gauge transformation was

\[
A_\mu \rightarrow A_\mu + \partial_\mu \Lambda,
\]

where \( \Lambda \) was the gauge parameter.

**Gauge Invariance** - The freedom to choose any gauge parameter and still be left with the theory unchanged.

**Fixing the Gauge** - The act of eliminating a degree of freedom by choosing a specific value for the gauge parameter(s). In the EM case, we picked a specific \( \Lambda \) to set \( \partial_\nu A^\nu = 0 \).

### 10.3 Manifolds

**Manifold** - An \( n \) dimensional manifold is a space that may be complicated globally, but locally, resembles \( \mathbb{R}^n \). This is accomplished by smoothly patching together several regions that look like \( \mathbb{R}^n \).

**Map** - A map is a generalization of a function between manifolds. Mathematically, a \( M \rightarrow N \) map is a relationship that assigns each element of manifold \( M \) to exactly one element from manifold \( N \).

**Smooth maps** - A map that is continuous and infinitely differentiable, also written as \( C^\infty \) maps.

**Invertible maps** - A map, whose inverse mapping still acts as a function.

**Pullback** - Suppose we have two manifolds \( M \) and \( N \) and that there exists a mapping, \( \phi : M \rightarrow N \), and a function \( f : N \rightarrow \mathbb{R} \). The pullback of \( f \), \( \phi^* f \), is simply the composition of \( f \) with \( \phi \):

\[
\phi^* f = (f \circ \phi).
\]  

**Pushforward** - For a vector, \( V(p) \) at point \( p \) on manifold \( M \), the pushforward vector, written as \( \phi_* V \), at the point \( \phi(p) \) on the manifold \( N \) is given by its action on the functions on \( N \):

\[
(\phi_* V)(f) = V(\phi^* f).
\]

### 10.4 Diffeomorphism

**Diffeomorphism** - For two given manifolds, \( M \) and \( N \), a diffeomorphism is an invertible mapping \( \phi : M \rightarrow N \), such that \( \phi \) and \( \phi^{-1} \) are both \( C^\infty \) mappings (ie, the mappings are continuous and infinitely differentiable).
Diffeomorphisms on the same manifold \((M \rightarrow M)\) - Any mapping \(\phi : M \rightarrow M\) is obviously invertible, thus provided that \(\phi\) is a \(C^\infty\) mapping, then \(\phi\) is a diffeomorphism.

Lie derivatives (Scalar fields) - The Lie derivative for a scalar field \(\phi\), with respect to a field \(\xi\) is given by

\[
\mathcal{L}_\xi \phi = \xi^\alpha \partial_\alpha \phi. \tag{204}
\]

Lie bracket - For two vectors fields \(X\) and \(Y\), the Lie bracket is the commutator given by:

\[
[X, Y] \mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu. \tag{205}
\]

Lie derivatives (Vector fields) - A Lie derivative is a derivative that measures the change of one field, \(U\), with respect to a vector field, \(V\), in a manifold. Mathematically speaking, the Lie derivative of a vector field \(U^\mu(x)\), along the vector field \(V\) is given by the Lie bracket \([V, U]\):

\[
\mathcal{L}_V U^\mu = [V, U] \mu. \tag{206}
\]

Lie derivatives (Tensor fields) - Extending the vector field definition, the Lie derivative of an arbitrary tensor field, \(T^{\mu_1 \mu_2 ... \mu_k}_{\nu_1 \nu_2 ... \nu_l}\), is given by:

\[
\mathcal{L}_V T^{\mu_1 \mu_2 ... \mu_k}_{\nu_1 \nu_2 ... \nu_l} = V^\sigma \partial_\sigma T^{\mu_1 \mu_2 ... \mu_k}_{\nu_1 \nu_2 ... \nu_l}
- (\partial_\nu V^{\mu_1}) T^{\mu_2 ... \mu_k}_{\nu_1 \nu_3 ... \nu_l}
- (\partial_\nu V^{\mu_2}) T^{\mu_1 \mu_3 ... \mu_k}_{\nu_1 \nu_2 ... \nu_l}
- ... \tag{207}
+ (\partial_\nu V^{\nu_1}) T^{\mu_1 \mu_2 ... \mu_k}_{\nu_2 \nu_3 ... \nu_l}
+ (\partial_\nu V^{\nu_2}) T^{\mu_1 \mu_2 ... \mu_k}_{\nu_1 \nu_3 ... \nu_l}
+ ... \]

Virtual Diffeomorphism - Reparametrizations that come up in the BB model that appear to have the form of an actual diffeomorphism, but in the end are not defined mathematically correct.

10.5 SSB

Spontaneous Symmetry Breaking - The concept of choosing a preferred frame when a given system has two or more mathematically equivalent coordinate frames, and as a result, we lose the symmetry of the theory.

Symmetry - Certain transformations that leaves the dynamics of a system unchanged, are called symmetries of the theory. Otherwise known as invariance in a theory.

Vacuum - Ground state in the potential

Vacuum Expectation Value - The expectation value of the ground state, often abbreviated as vev.

Goldstone's Theorem - For any spontaneously broken global symmetry, Goldstone's theorem states that we should find massless modes.
Nambu-Goldstone Modes - The massless modes that result from spontaneously breaking a global symmetry, often abbreviated as NG modes.

Higgs Mechanism - Occurs when we go from a global transformation to a local one. The Higgs mechanism is the process by which gauge fields of a theory acquire mass, involving the "eating" of massless NG modes found from the global symmetry breaking.

Gauge-Covariant Derivatives - Modified derivative necessary to keep covariance in local gauge transformations. The gauge-covariant derivative for SO(2) and U(1) is defined as

\[ D_\mu = \partial_\mu + igA_\mu. \]
References


