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Chow's Theorem

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1 Introduction

In 1949, W.-L. Chow published the paper *On Compact Complex Analytic Varieties* in which he proved the theorem that now bears his name

Theorem 1.0.1 (Chow). *Every closed analytic subspace of $\mathbb{P}^n_{\mathbb{C}}$ is an algebraic set.*

In that paper, he gave a completely analytic proof of this result. His proof, as opposed to the one J.-P. Serre gave as a consequence his famous correspondence theorem GAGA, did not involve the scheme-theoretic or homological algebraic methods.

In 1956, a year after he published the paper *Faisceaux algebriques coherents*, or *Coherent algebraic sheaves*, Serre proved the GAGA correspondence theorem; the theorem takes its name from the paper—*Géometrie Algébrique et Géométrie Analytique*. The GAGA correspondence states that an equivalence of categories that preserves cohomology exists between coherent sheaves on a complex projective space X and on its associated analytic space X^h.

Among other consequences of GAGA that bridge complex algebraic geometry and complex analytic geometry is Chow's theorem. The subject of this thesis is the proof of Chow's theorem using GAGA. We will introduce the necessary sheaf theory, scheme theory and complex analysis background before stating GAGA and proving Chow's theorem.

Our rings are commutative with a unity.

2 Sheaves and Ringed Spaces

2.1 Sheaves

Sheaves are tools that allow us to track local algebraic structures of topological spaces. We will begin by defining presheaves.

2.1.1. Presheaves.

Let X be a topological space.

Definition 2.1.1. A *presheaf* $\mathscr{F}$ of abelian groups on X consists of the following data:

1. For every open set $V \subset X$, $\mathscr{F}(V)$ is an abelian group and

2. If V, U are open subsets of X such that $V \subset U$, then we have a *restriction morphism* $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$,

such that

(a) $\mathscr{F}(\varnothing) = 0$ (zero group).

(b) $\rho_{UU} = 1$

(c) If $V \subset U \subset W$ are open in X, then $\rho_{WV} = \rho_{UV} \circ \rho_{WU}$.

In other words,

Definition 2.1.2. A presheaf $\mathscr{F}$ of abelian groups over X is a contravariant functor from the category of Top(X) to the category of abelian groups Ab, where the open subsets of X are the objects and inclusion maps are the morphisms of Top(X); moreover, Hom(V, U) is empty if $V \not\subset U$ and is a singleton (an inclusion homomorphism) if $V \subset U$.

Definition 2.1.3. If $\mathscr{F}$ is a *presheaf on* X and $x \in X$, the *stalk* $\mathscr{F}_x$ of $\mathscr{F}$ at x is the direct limit via the restriction maps ρ of the groups $\mathscr{F}(U)$, where U are open neighborhoods of x.

Elements of the abelian group $\mathscr{F}(U)$ for $U \subset X$ are known as *sections* of the presheaf $\mathscr{F}$ over U. So, $\mathscr{F}(U)$, or as it is sometimes denoted $\Gamma(U, \mathscr{F})$ is a group of sections of $\mathscr{F}$ over U. Finally, a common shorthand for $\rho_{UV}(s)$ is $s|_V$, as $s \in \mathscr{F}(U)$.

2.1.2. **Morphisms of presheaves.** Let $\mathscr{F}, \mathscr{G}$ be two presheaves on X. A *morphism of presheaves* $\psi : \mathscr{F} \to \mathscr{G}$ consists of, for every open subset U of X, a group homomorphism $\psi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ which is compatible with the restrictions ρ_{UV}.

The map ψ is *injective* if for every open subset U of X, $\psi(U)$ is injective. On the other hand, ψ is *surjective* if for any open subset U of X, $s \in \mathscr{F}(U)$, and $x \in U$, the canonically induced group homomorphism $\psi_x : \mathscr{F}_x \to \mathscr{G}_x$ that maps $(\psi(U)(s))_x$ to $\psi_x(s_x)$ is surjective. But ψ is an *isomorphism* if $\psi(U)$ is an isomorphism for every open subset $U \subset X$.

2.1.3. **Sheaves.** Putting certain conditions on the sections of a presheaf gives rise to sheaves; precisely, sheaves are presheaves with extra data on their sections. As we will see in this section, sheaves are more useful than presheaves in telling us about local data.

Definition 2.1.4. A *sheaf* $\mathscr{F}$ of abelian groups on X is a presheaf with the following properties.

1. (Locality) Let $\{V_i\}$ be an open cover of U. Suppose that $s \in \mathscr{F}(U)$ and that we have $s|_{V_i} = 0$ for all i. Then $s = 0$.

2. (Gluing) If $\{V_i\}$ is an open cover of U, and suppose $s_i \in \mathscr{F}(V_i)$. Also suppose that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all i, j. Then, there exists $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$, for all i.

Example 2.1.5. (1) The constant sheaf $\underline{\mathbf{Z}}$ of sets on any topological space X where every stalk is $\mathbf{Z}$, and every homomorphism is the identity.

(2) The zero sheaf is the constant sheaf $\underline{0}$, where 0 denotes the trivial additive group.

We will give an example of a presheaf that is not a sheaf.

Examples 2.1.6. Consider the space $X = \{x, y\}$ with the discrete topology. Let the presheaf $\mathscr{F}$ be defined as follows: $\mathscr{F}(\varnothing) = \{\varnothing\}, \mathscr{F}(\{x\}) = \mathscr{F}(\{y\}) = \mathbf{R}$, and $\mathscr{F}(\{x, y\}) = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$. Define the restriction in the obvious way, then we have a presheaf. But although we can glue any two sections over $\{x\}$ and $\{y\}$, we can't glue them uniquely.

2.1.4. **Morphisms of sheaves.** A morphism of sheaves is just a morphism of presheaves. However, in the case of sheaves,

Proposition 2.1.7. *If $\psi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves on X. Then ψ is an isomorphism of sheaves if and only if ψ_x is an isomorphism for every $x \in X$.*

Proof. See PROP 2.12 in [Liu06]. □

This is in general false for presheaves. That is, sheaves are more descriptive of what is going on locally than presheaves.

Definition 2.1.8. Let $f : X \to Y$ be a continuous mapping of topological spaces. Then the *direct image functor* f_* sends a sheaf $\mathscr{F}$ on X to its direct image presheaf $f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U))$ which is in fact a sheaf on Y.

Definition 2.1.9. A *subsheaf* of a sheaf $\mathscr{F}$ is a sheaf $\mathscr{F}'$ such that for every open set $U \subset X$, $\mathscr{F}'(U)$ is a subgroup of $\mathscr{F}(U)$, and the restriction maps of the sheaf $\mathscr{F}'$ are induced by those of $\mathscr{F}$. If $\mathscr{G}$ is another sheaf on X, the *quotient sheaf* $\mathscr{F}/\mathscr{G}$ is the associated[1] sheaf to the presheaf $U \mapsto \mathscr{F}(U)/\mathscr{G}(U)$ for all open $U \subset X$.

Proposition 2.1.10. *Let $\mathscr{F}, \mathscr{G}$ be sheaves, and let ψ be a morphism from $\mathscr{F}$ to $\mathscr{G}$. For all $x \in X$, let $\mathscr{N}_x$ be the kernel and $\mathscr{I}_x$ the image of ψ_x. Then $\mathscr{N} = \bigcup \mathscr{N}_x$ is a subsheaf of $\mathscr{F}$, $\mathscr{I} = \bigcup I_x$ is a subsheaf of $\mathscr{G}$ and ψ defines an isomorphism of $\mathscr{F}/\mathscr{N}$ and $\mathscr{I}$.*

Proof. See PROP. 7 in §8 [Ser55]. □

We call the sheaf $\mathscr{N}$ above the *kernel* of ψ, $\mathscr{I}$ the *image* of ψ, and the quotient $\mathscr{G}/I$ the *cokernel* of ψ.

Definition 2.1.11. Given a sheaf morphism $\psi : \mathscr{F} \longrightarrow \mathscr{G}$, we say ψ is injective if $\ker \psi = 0$, surjective if $\operatorname{coker} \psi = 0$, and bijective if it is both injective and surjective.

Definition 2.1.12. If U is an open subset of X every sheaf $\mathscr{F}$ on X induces a subsheaf $\mathscr{F}|_U$ on U in an obvious way, where $\mathscr{F}|_U(V) = \mathscr{F}(V)$ for every open $V \subset U$. We call $\mathscr{F}|_U$ the *restriction of $\mathscr{F}$ to U.*

Another way to characterize the sheaf conditions stated earlier is as follows.

Proposition 2.1.13. *The presheaf $\mathscr{F}$ over the topological space X is a sheaf if and only if, for every open subset $V \subset X$ and any open cover $V = \bigcup_{i \in I} U_i$, the following sequence*

$$0 \longrightarrow \mathscr{F}(V) \xrightarrow{\ \alpha\ } \prod_{i \in I} \mathscr{F}(U_i) \xrightarrow{\ \beta\ } \prod_{i,j \in I} \mathscr{F}(U_i \cap U_j)$$

[1] that is, identifying the sheaf of sections of an appropriate topological space (see *sheafification* §II.1 PROP-DEFN 1.2[Har97]).

where α takes $s \in \mathscr{F}(V)$ to the product of its restrictions, and

$$\beta : \{s_i\}_{i\in I} \mapsto \{\rho_{U_i,U_i\cap U_j}(s_i) - \rho_{U_j,U_i\cap U_j}(s_j)\}_{i,j\in I}$$

is exact.

Proof. *Omitted.* □

2.2 Ringed Spaces

Definition 2.2.1. A *ringed space* is a pair $(X, \mathscr{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathscr{O}_X$ on X.

We call the sheaf the *structure sheaf* of the space.

Definition 2.2.2. A *locally ringed space* is a ringed space $(X, \mathscr{O}_X)$ such that for each point $x \in X$, the stalk $\mathscr{O}_x$ is a local ring.

Many types of spaces can be identified as certain types of ringed spaces.

Examples 2.2.3. (1) Let X be a topological space. Then there is a corresponding ringed space $(X, \mathscr{C}_X)$ where $\mathscr{C}_X$ is the sheaf of germs of continuous functions on X.

(2) If X is an algebraic variety carrying the Zariski topology (§3.1.1), we can define a locally ringed space by taking $\mathscr{O}_X(U)$ to be the ring of rational mappings defined on the Zariski-open set U that don't become infinite within U.

(3) Schemes are locally ringed spaces that are locally isomorphic to the spectrum of a ring; see §3.

(4) A *complex analytic space* (DEFINITION.4.1.1) is a locally ringed space whose structure is a $\underline{\mathbf{C}}$-algebra, where $\underline{\mathbf{C}}$ is the constant sheaf on a topological space with value $\mathbf{C}$.

2.2.1. **Morphisms of ringed spaces.**

Definition 2.2.4. A *morphism of ringed spaces*

$$(\psi, \psi_\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$$

consists of a continuous map $\psi : X \to Y$ and a morphism of sheaves of rings $\psi^\# : \mathscr{O}_Y \to \psi_*\mathscr{O}_X$. A morphism $(\psi, \psi^\#)$ is an isomorphism if and only if ψ is a homeomorphism of the underlying topological spaces, and $\psi^\#$ is an isomorphism of the sheaves.

Definition 2.2.5. A *morphism of locally ringed spaces* $(X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces $(\phi, \phi^\#)$ such that for all $x \in X$ the induced homomorphism on stalks

$$\phi_x^\# : (\phi^{-1}\mathscr{O}_Y)_x = \mathscr{O}_{Y,\phi(x)} \longrightarrow \mathscr{O}_{X,x}$$

is a local ring homomorphism.

That it is a local homomorphism means that $\phi_x^{\#-1}(\mathfrak{m}_x) = \mathfrak{m}_{\phi(x)}$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathscr{O}_{X,x}$.

The composition of two morphisms of locally ringed spaces is again a morphism of locally ringed spaces. Therefore locally ringed spaces form a category.

Example 2.2.6. Let X be a topological space and consider the sheaf $\mathscr{C}_X$ of $\mathbf{R}$-valued continuous functions on X (i.e., for $U \subset X$ open, $\mathscr{C}_X(U)$ is the $\mathbf{R}$-algebra of continuous functions $s : U \to \mathbf{R}$). For $x \in X, \mathscr{C}_{X,x}$ is the ring of germs $[s]$ of continuous functions s in a neighborhood of x. Let $\mathfrak{m}_x \subset \mathscr{C}_{X,x}$ be the set of germs $[s]$ such that $s(x) = 0$. This is a unique maximal ideal and $(X, \mathscr{C}_X)$ is a locally ringed space [UG10].

Lastly,

Definition 2.2.7. A morphism $(\psi, \psi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is an *open immersion* (resp. *closed immersion*) if f is a topological open immersion (resp. closed immersion) and if $f_x^\#$ is an isomorphism (resp. if $f_x^\#$ is surjective) for every $x \in X$.

Moreover, if V is open in Y, the restriction $(V, \mathscr{O}_Y|_V)$ is a ringed space. So, when we consider an open subset, it will inherit this structure induced by $\mathscr{O}_Y$. Hence, $(\psi, \psi^\#)$ is an open immersion if and only if there exists an open subset V of Y such that $(\psi, \psi^\#)$ induces an isomorphism from $(X, \mathscr{O}_X)$ onto $(V, \mathscr{O}_Y|_V)$.

2.3 Sheaves of Modules

These are special types of sheaves that we almost always work with in this paper.

2.3.1. **Definitions.**

Definition 2.3.1. Let $(X, \mathscr{O}_X)$ be a ringed space. A *sheaf of $\mathscr{O}_X$-modules* is a sheaf $\mathscr{F}$ on X such that $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module for every open subset U, and that if $V \subset U$, then $(st)|_V = s|_V t|_V$ for every $s \in \mathscr{O}_X(U)$ and every $t \in \mathscr{F}(U)$.

Examples 2.3.2. (1) If $\mathscr{O}$ is the constant sheaf $\underline{\mathbf{Z}}$, then the sheaf of $\mathscr{O}$-modules on it is the same as the sheaf of the abelian groups $\underline{\mathbf{Z}}$.

(2) If X is a smooth variety, then the cotangent sheaf is a sheaf of modules.

Definition 2.3.3. An *$\mathscr{O}_X$-submodule* $\mathscr{G}$ of $\mathscr{F}$ is an $\mathscr{O}_X$-module $\mathscr{F}$ such that $\mathscr{F}(U)$ is a subset of $\mathscr{G}(U)$ for all open sets $U \subset X$ and such that the inclusions $\iota_U : \mathscr{F}(U) \hookrightarrow \mathscr{G}(U)$ form a homomorphism $\iota : \mathscr{F} \longrightarrow \mathscr{G}$ of $\mathscr{O}_X$-modules.

The $\mathscr{O}_X$-submodules of $\mathscr{O}_X$ are called *ideals of* $\mathscr{O}_X$.

2.3.2. **Operations on sheaves of modules.**

Let $\mathscr{O}$ be a sheaf of rings on X and $\mathscr{N}, \mathscr{M}$ be two sheaves of $\mathscr{O}$-modules.

Definition 2.3.4. The *direct sum* $\mathscr{N} \oplus \mathscr{M}$ is the sheaf of $\mathscr{O}$-modules defined by $(\mathscr{N} \oplus \mathscr{M})(U) = \mathscr{N}(U) \oplus \mathscr{M}(U)$.

By recurrence, the definition of direct sum extends to a finite number of $\mathscr{O}$-modules; if we have n sheaves isomorphic to one sheaf $\mathscr{F}$, we denote their direct sum by $\mathscr{F}^n$.

Definition 2.3.5. An $\mathscr{O}_X$-module $\mathscr{F}$ on a ringed space X is called *locally free of finite rank* if every point in X has an open neighborhood U such that the restriction $\mathscr{F}|_U$ is isomorphic to a finite direct sum of copies of $\mathscr{O}_X|_U$.

Definition 2.3.6. The *tensor product* $\mathscr{N} \otimes \mathscr{M}$ is the sheaf of $\mathscr{O}$-modules associated to the presheaf $U \mapsto \mathscr{N}(U) \otimes_{O(U)} \mathscr{M}(U)$.

2.3.3. **Support of an $\mathscr{O}_X$-module.**

Definition 2.3.7. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then

$$\mathrm{Supp}(\mathscr{F}) = \{x \in X : \mathscr{F}_x \neq 0\}$$

is called the *support of* $\mathscr{F}$.

2.3.4. **Coherent Sheaves.** Coherent sheaves are generalizations of vector bundles. But unlike vector bundles, the category of coherent sheaves on a topological space X forms an abelian category. So we can take kernels, images, and cokernels.

Let $(X, \mathscr{O}_X)$ be a ringed space, and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. We say that $\mathscr{F}$ is *finitely generated* if for every $x \in X$, there exists an open neighborhood U of x, an integer $n \geq 1$, and a surjective homomorphism $\mathscr{O}_X^n|_U \longrightarrow \mathscr{F}|_U$.

On the other hand, we say $\mathscr{F}$ is *finitely presented* if in addition there exists an integer $m \geq 1$, and a morphism of sheaves $\mathscr{O}_X^m|_U \to \mathscr{O}_X^n|_U$ such that the sequence

$$\mathscr{O}_X^m|_U \longrightarrow \mathscr{O}_X^n|_U \longrightarrow \mathscr{F}$$

is exact.

Definition 2.3.8. The sheaf $\mathscr{F}$ is *coherent* if it is finitely generated, and if for every open subset U of X, and for every homomorphism $\psi : \mathscr{O}_X^n|_U \longrightarrow \mathscr{F}|_U$, the kernel $\ker \psi$ is finitely generated.

Examples 2.3.9. (1) If X is a complex analytic variety, the sheaf of germs of holomorphic functions on X is a coherent sheaf of rings, by Oka's Coherence Theorem (Theorem 6.4.1[Hor90]).

(2) If X is an algebraic variety, the sheaf of local rings of X is a coherent sheaf of rings.

Coherent sheaves are actually a specific type of a more general class of sheaves of modules called *quasi-coherent sheaves*.

Definition 2.3.10. A sheaf $\mathscr{G}$ of $\mathscr{O}_X$-modules is called *quasi-coherent* if it is, locally, isomorphic to the cokernel of a map between free $\mathscr{O}_X$-modules.

It is apparent that coherence is a local property as the sheaf $\mathscr{F}$ is coherent if and only if $\mathscr{F}|_{U_\lambda}$ is coherent for every $\lambda \in \Lambda$, where $\{U_\lambda | \lambda \in \Lambda\}$ is an open cover of X. We observe a few more properties

Proposition 2.3.11. —

1. *Any subsheaf of finite type of a coherent sheaf is coherent.*

2. *Let* $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{K} \to 0$ *be an exact sequence of homomorphisms. If two of the sheaves* $\mathscr{F}, \mathscr{G}, \mathscr{K}$ *are coherent, so is the third.*

3. *Let ϕ be a homomorphism from a coherent sheaf $\mathscr{F}$ to a coherent sheaf $\mathscr{G}$. The kernel, the cokernel and the image of ϕ are also coherent sheaves.*

Proof. —

1. NO. 12, PROP. 3[Ser55]
2. NO. 13, THEOREM 1 [Ser55]
3. Because $\mathscr{F}$ is coherent, the image is of finite type and by (1) coherent; and applying (2) to the following exact sequences

$$0 \to \ker\phi \to \mathscr{F} \to \operatorname{Im}\phi \to 0,$$

$$0 \to \operatorname{Im}\phi \to \mathscr{F} \to \operatorname{coker}\phi \to 0,$$

we get that so are the cokernel and the kernel of ϕ. □

The key property of modules of finite type is highlighted by the following proposition.

Proposition 2.3.12. *Let $(X, \mathscr{O}_X)$ be a ringed space and let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type. Let $x \in X$ be a point and let $s_i \in \mathscr{F}(U)$ for $i = 1, \ldots, n$ be sections over some open neighborhood of x such that the germs $(s_i)_x$ generate the stalk $\mathscr{F}_x$. Then there exists an open neighborhood $V \subset U$ such that the $s_i|_V$ generate $\mathscr{F}|_V$.*

Proof. PROP. 7.29, [UG10] □

The following result applied to coherent sheaves will be crucial in the proof of Chow's theorem.

Corollary 2.3.13. *Let $(X, \mathscr{O}_X)$ be a ringed space and let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type. Then* $\operatorname{Supp}(\mathscr{F})$ *is closed in X.*

3 Schemes

Schemes were introduced by Alexander Grothendieck. They generalize the notion of variety. They are constructed locally from objects called *affine schemes*. An affine scheme is a ringed space; so, we will start by specifying the topological space before we construct a structure sheaf on it.

3.1 Affine Schemes

3.1.1. **Spec.** The *spectrum* Spec R of a ring R is, as a space, the set of all prime ideals of R; as a topological space, it carries the *Zariski topology.* Declare the closed sets to be sets of the form $\mathbb{V}(I) = \{\mathfrak{p} \supset I \mid \mathfrak{p} \in \mathrm{Spec}\,(R)\}$, where I is an ideal in R.

To see that this actually defines a topology, first note that $\varnothing = \mathbb{V}(\mathrm{Spec}\,(R))$ and $\mathrm{Spec}\,(R) = \mathbb{V}(\varnothing)$, while for any arbitrary collection of closed sets indexed by I,

$$\bigcap_{\iota \in I} V(\mathfrak{a}_\iota) = V(\mathfrak{b}),$$

where $\mathfrak{b}$ is the ideal generated by the union of a_ι. Furthermore, for any closed $V(\mathfrak{a})$, $V(\mathfrak{b})$,

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}).$$

Hence, we have a topology on $\mathrm{Spec}\,(R)$ as required.

If we let $X = \mathrm{Spec}\,(R)$ and Rf be the ideal generated by $f \in R$, then the set

$$X_f = X - V(Rf)$$

for any f is open in X. These X_f in fact make an open base for the Zariski topology and are referred to as the distinguished open sets of this topology.

An interesting property of this topology is that it is not Hausdorff. The set $\mathfrak{p} \in \mathrm{Spec}\,(R)$ is Z-closed if and only if $\mathfrak{p}$ is maximal. For instance, if $R = k[X]$, where k is a field, we get $\mathrm{Spec}\,(R) = \mathbb{A}^1_k$; but the point 0 is not Z-closed because $\{0\}$ is not a maximal ideal.

3.1.2. **The sheaves $\widetilde{M}$.** Let R be a ring and $X = \mathrm{Spec}\,(R)$. Also suppose that M is an R-module. We will now define a canonical sheaf of modules on X. The case of $M = R$ reduces to $\widetilde{M} = \mathscr{O}_{\mathrm{Spec}(R)}$.

Recall that the distinguished open sets X_f for $f \in R$ form a basis for the Zariski topology on X. To define a sheaf on X, then it suffices to work with the X_f's. To see this, let $\mathscr{B}$ be a base of open subsets on X and let $U = \cup U_i$, where $U_i \in \mathscr{B}$. Then $\mathscr{A}(U)$ contains elements $(s_i)_i \in \prod_i \mathscr{A}_0(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|U_i \cap U_j$, where $\mathscr{A}$ is the extension of a $\mathscr{B}$−sheaf $\mathscr{A}_0$. Furthermore, this extension is unique up to isomorphism.

Let $M_f = M \otimes_R R_f$ where R_f is the localization of R at f. That is, M_f is the set of symbols $m/f^n, m \in M, n \in \mathbf{Z}$ modulo the relation that $m_1/f^{n_1} = m_2/f^{n_2}$

if and only if $f^{n_2+k} \cdot m_1 = f^{n_1+k} \cdot m_2$ for some $k \in \mathbf{Z}$. We claim that this defines a presheaf on X.

It only remains to show the existence of restriction functions. Suppose $X_g \subset X_f$. This implies that there exists $m \geq 1$ and $b \in R$ such that $g^m = fb$. As f is then invertible, we have a restriction homomorphism

$$R_f \to R_g : af^{-n} \mapsto (ab^n)g^{-mn}.$$

In fact, $\widetilde{M}$ is also a sheaf. We will reproduce the proof from [Liu06]:p.42. Let $X_{f_i} = X_i$. Then

$$V\left(\sum_i f_i R\right) = \bigcap_i V(f_i R) = \varnothing,$$

and $R = \sum_i f_i R$. Hence, there exists a finite subset $F \subset I$ such that $1 \in \sum_{i \in F} f_i R$.

To check for Definition 1.1.4 (1), suppose $s \in R$ and $s|V_i = 0$ for all i. Then there exists an $m \geq 1$ such that $f_i^m s = 0$ for every $i \in F$. Then $X = \bigcup_{i \in F} D(f_i) = \bigcup_{i \in F} D(f_i^m)$. It follows that $1 \in \sum_{i \in F} f_i^m R$ and so,

$$s \in \sum_{i \in F} s f_i^m R = \{0\}$$

as required.

To check for Definition 1.1.4 (2), suppose $s_i \in \widetilde{M}(U_i)$ be sections that agree on the intersections $U_i \cap U_j$. Then, there exists an $m \geq 1$ such that $s_i = b_i f_i^{-m} \in R_{f_i}$ for every $i \in F$. But since $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there exists an $r \geq 1$ such that $(b_i f_j^m - b_j f_i^m)(f_i f_j)^r = 0$ for every $i, j \in F$. As above, $1 = \sum_{j \in F} a_j f_j^{m+r}$ with $a_j \in R$. Let us set $s = \sum_{j \in F} a_j b_j f_j^r \in R$. For every $i \in F$, we have

$$f_i^{m+r} s = \sum_{j \in F} a_j b_j f_j^r f_i^{m+r} = \sum_{j \in F} a_j b_i f_i^r f_j^{m+r} = b_i f_i^r.$$

Hence, $s|_{U_i} = s_i$ for every $i \in F$. That is, if $i \in I$, then $(s|_{U_i})|_{U_i \cap U_j} = s_i|_{U_i \cap U_j}$ for every $j \in F$. Since $U_i = \bigcup_{j \in F} U_i \cap U_j$, we have $s|_{U_i} = s_i$. This concludes the proof that $\widetilde{M}$ is a sheaf.

Definition 3.1.1. An *affine scheme* is a locally ringed space isomorphic as a locally ringed space to $(\operatorname{Spec}(R), \widetilde{R})$ for some ring R.

3.2 General Schemes

Definition 3.2.1. A *scheme* is a ringed topological space $(X, \mathscr{O}_X)$ admitting an open covering $\{U_i\}_i$ such that $(U_i, \mathscr{O}_X|_{U_i})$ is an affine scheme for every i.

Remark 3.2.2. In other words, A scheme is a ringed space that is *locally affine*. If $(X, \mathscr{O}_X)$ is a ringed space and X is covered by open sets U_i, then $(X, \mathscr{O}_X)$ is locally affine if there exist rings R_i and homeomorphisms $\operatorname{Spec}(R_i) \cong U_i$ with $\mathscr{O}_X|_{U_i} \cong \mathscr{O}_{\operatorname{Spec}(R_i)}$ As a consequence, we have that (1) $\mathscr{O}_{\operatorname{Spec}(R_i)}(X_f) = R[f^{-1}]$; (2) The stalks $\mathscr{O}_x$ are local rings, and (3) X and $\operatorname{sp}(\operatorname{Spec}(\mathscr{O}(X)))$ are homeomorphic.

Proposition 3.2.3. *Let X be a scheme*[2]*. Then for any open subset U of X, the ringed space $(U, \mathscr{O}_X|_U)$ is also a scheme.*

Proof. PROP 3.9 - [DE07] □

3.2.1. Maps between schemes.

Definition 3.2.4. A morphism or map between schemes X and Y is a pair $(\psi, \psi^\#)$, where $\psi : X \to Y$ is a continuous map on the underlying topological spaces and

$$\psi^\# : \mathscr{O}_Y \to \psi_* \mathscr{O}_X$$

is a map of sheaves on Y satisfying the condition that for any point $p \in X$ and any neighborhood U of $q = \psi(p)$ in Y, a section $f \in \mathscr{O}_Y(U)$ vanishes at q if and only if the section $\psi^\# f$ of

$$\psi_* \mathscr{O}_X(U) = \mathscr{O}_X(\psi^{-1} U)$$

vanishes at p.

3.2.2. Subschemes.

We are particularly interested in the closed ones; so we will first define

Definition 3.2.5. A *morphism of schemes $\phi : Y \to X$* is a closed embedding if every point $x \in X$ has an affine neighborhood U such that $\phi^{-1}(U) \subset Y$ is an affine subscheme and the homomorphism $\psi_U : \mathscr{O}_X(U) \longrightarrow \mathscr{O}_Y(\phi^{-1}(U))$ is surjective.

In this case, Y is said to be a *closed subscheme* of X.

[2] the ringed space $(X, \mathscr{O}_X)$ by abuse of notation

3.2.3. **S-Schemes.**

Definition 3.2.6. Let S be a fixed scheme. A *scheme over* S is a scheme X, together with a morphism $X \to S$. If X and Y are schemes over S, a morphism of X to Y as schemes over S, called an *S-morphism* is a morphism $\phi : X \to Y$ which is compatible with the given morphisms to S.

3.3 Projective Schemes

Both GAGA and Chow's Theorem are about projective spaces. Unlike the section on affine schemes, where we built up them from the ground up, this time, to enrich our discussion of projective spaces, we will define projective schemes as subschemes in a projective space. Another approach is to construct projective schemes from graded algebras. This is done by using the Proj functor. We will also demonstrate this in the end.

3.3.1. **Projective n-Space.** We will be working in the field $\mathbf{C}$. For this construction we will closely follow [Har97].

Definition 3.3.1. The set of equivalence classes of $(n+1)$-tuples $(a_0, \dots, a_n)$ of elements of $\mathbf{C}$, not all zero, under the equivalence relation given by

$$(a_0, \dots, a_n) \sim (\lambda a_0, \dots, \lambda a_n)$$

for all $\lambda \in \mathbf{C}$, $\lambda \neq 0$ is called the *complex projective n-space* and it is denoted by $\mathbb{P}^n$. Unless otherwise stated, $\mathbb{P}^n$ refers to $\mathbb{P}^n_{\mathbf{C}}$.

Equivalently, $\mathbb{P}^n$ is the set of all one-dimensional subspaces of $\mathbb{A}^{n+1}$. Or in other words, it is the quotient of the set $\mathbb{A}^{n+1} - \{(0, \dots, 0)\}$ under the equivalence relation which identifies points lying on the same line through the origin.

We call these equivalence classes points of $\mathbb{P}^n$ and the elements of an equivalence class is referred to as a *set of homogeneous coordinates for* the class.

Now that we have defined the main subject, we remind the reader that GAGA tells us that analytic (or holomorphic) objects on the complex projective space $\mathbb{P}^n$ are indeed algebraic. This hints that projective spaces are unique and important as they seem bridge the gap between algebraic and complex geometry.

Definition 3.3.2. A subset Y of $\mathbb{P}^n$ is an *algebraic set* if there exists a set T of homogeneous elements of $\mathbf{C}[x_0, \dots, x_n]$ such that Y is the vanishing set of T.

3.3.2. **Structure of $\mathbb{P}^n$.** Just like the affine space, we can make it into a topological space by endowing it the Zariski topology; take the open sets to be the complements of the algebraic sets.

Definition 3.3.3. A *projective variety* is a zero locus of a finite set of homogeneous polynomials, with the Zariski topology, the closed sets being projective algebraic sets.

Proposition 3.3.4. *$\mathbb{P}^n$ is a compact Hausdorff space with the quotient topology (analytic topology).*

Proof. [RCG65]:p.150. □

3.3.3. **Projective Schemes.**

Definition 3.3.5. A scheme isomorphic to a closed subscheme of $\mathbb{P}^n$ is a *projective scheme* over **C**.

One can obtain a closed subscheme $X \subset \mathbb{P}^n$ by glueing $n+1$ affine schemes $V_i = U_i \cap X$ for $i = 0, \dots, n$, where $V_i = X \cap \mathbb{A}^n_i$; the structure sheaf of V_i is the restriction $\mathscr{O}_X|_V i$. That is, $V_i = \operatorname{Spec}(C)_i$ where $C_i = R_i/\mathfrak{a}_i$ with $R_i = R[X_0/X_i, \dots, X_n/X_i]$ and $\mathfrak{a}_i$ is an ideal of R_i. More generally,

Definition 3.3.6. If R is a ring, a *projective scheme* over R is an R-scheme that is isomorphic to a closed subscheme of $\mathbb{P}^n_R$ for some $n \geq 0$.

The main difference between algebraic and projective varieties is that the latter are compact over **C** in the usual topology. That is, if $f : X \to Y$ is a morphism of projective varieties, then $f(X)$ is a closed subspace of Y. More generally,

Proposition 3.3.7. *Every complex projective scheme is compact in the complex topology.*

Proof. THEOREM 13.40, [UG10]. □

3.3.4. **Proj.** The functor Proj gives an alternative way of producing a projective scheme (from a graded ring.) The construction is analogous to the Spec construction, where we produced an affine scheme from a commutative ring. In contrast, the points of the space from Proj are homogeneous prime ideals of R that do not contain the irrelevant ideal.

For convenience, we will first recall a few definitions.

Definition 3.3.8. A *graded ring* is a ring S, together with a decomposition $S = \bigoplus_{d \geq 0} S_d$ of S into the direct sum of abelian groups S_d, such that for any $d, e \geq 0$, $S_d \cdot S_e \subset S_{d+e}$.

An element of S_d is called a *homogeneous element of degree d*.

Definition 3.3.9. An ideal $\mathfrak{a} \subset S$ is a *homogeneous ideal* if

$$\mathfrak{a} = \bigoplus_{d \geq 0} (\mathfrak{a} \cap S_d).$$

We call

$$S_+ = \bigoplus_{d > 0} S_d$$

the *irrelevant* ideal.

Also recall that a homogeneous ideal $\mathfrak{p}$ is prime if and only if for any two homogeneous elements $a, b \in S$, $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

We are now ready to begin. Let S be a graded ring. We will first define the set $\operatorname{Proj} S$ and specify a topology before constructing the structure sheaf that makes it a projective scheme.

Define $\operatorname{Proj} S$ to be the set of all homogeneous prime ideals that do not contain the irrelevant ideal S_+. Next, we specify a topology on $\operatorname{Proj} S$ by setting the closed sets to be the sets of the form

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{a} \subset \mathfrak{p}\}$$

where $\mathfrak{a}$ is a homogeneous ideal of S. To see that this indeed defines a topology, let $\{\mathfrak{a}_i\}_{i \in I}$ be a family of homogeneous ideals of S, then we clearly have

$$\bigcap V(\mathfrak{a}_i) = V\left(\sum \mathfrak{a}_i\right).$$

On the other hand if we have $\mathfrak{a}$ and $\mathfrak{b}$ two homogeneous ideals in S, we have $V(\mathfrak{b}) \cup V(\mathfrak{a}) = V(\mathfrak{a}\mathfrak{b})$.

We will now attach to the space $\operatorname{Proj} S$ a structure of a sheaf $\mathscr{P}$. This sheaf will in the end give us a structure of a scheme on $\operatorname{Proj} S$. We will follow [DE07]. As we did (§3.1.2) for the sheaves $\widetilde{M}$, we will just specify the structure on each of a basis of open sets in the topology we just assigned.

Denote by X_f the open set $\operatorname{sp}(\operatorname{Proj} S) - V(\langle f \rangle)$, where f is any homogeneous element of S of degree 1. So, X_f does not contain f and S_+. We can identify the points of X_f with the homogeneous primes of $S[f^{-1}]$. In fact, we can identify X_f with $\operatorname{Spec}(S[f^{-1}]_0)$, where $S[f^{-1}]_0$ denotes the primes of the ring of elements of degree 0 in $S[f^{-1}]$. That is, we have an open affine subscheme of $\operatorname{Proj} S$ with the structure of the affine scheme we just identified. Denote this subscheme by $(\operatorname{Proj} S)_f$.

Now $\operatorname{Proj} S$ is a scheme via the affine open covering of $\operatorname{Proj} S$ given by open sets $(\operatorname{Proj} S)_{x_i} = \operatorname{Proj} S - V(x_i)$, where x_i are elements of degree 1 generating an ideal whose radical is S_+. To verify this we will check for the gluing condition; let $g \in S$ be another element of degree 1. Then, with the spectrum

$$S[f^{-1}]_0[(g/f)^{-1}] = S[f^{-1}, g^{-1}]_0,$$

the open set $(\operatorname{Proj} S)_f \cap (\operatorname{Proj} S)_g$ is an open affine subset of $(\operatorname{Proj} S)_f$. By symmetry of the expressions, we have verified that

$$((\operatorname{Proj} S)_f)_{(g/f)} = ((\operatorname{Proj} S)_g)_{(f/g)}.$$

Hence, $(\operatorname{Proj} S, \mathscr{P})$ is a scheme.

In fact, if $S = \mathbf{C}[x_0, \ldots, x_n]$ the polynomial ring graded by the degree of homogeneous polynomials, then $\mathbb{P}^n = \operatorname{Proj} C[x_0, \ldots, x_n]$ with

$$X_{x_i} = \operatorname{Spec}(R[x_0/x_i, \ldots, x_n/x_i])$$

for all $i = 0, \ldots, n$. More generally, we can redefine the projective n-space over a ring R.

Definition 3.3.10. If R is a ring, we define *projective n-space* over R to be the scheme $\mathbb{P}^n_A = \operatorname{Proj} R[X_0, \ldots, X_n]$.

4 Analytic Spaces

4.1 Definitions

If the coherent algebraic sheaf is defined on an algebraic variety X, then a coherent analytic sheaf is defined on the associated analytic space X^h of X. Recall that the algebraic variety X intrinsically comes with the Zariski topology; the same set with the usual topology is what we call the associated analytic space X^h. First we will define analytic varieties.

Definition 4.1.1. If $U \subset \mathbf{C}^n$ is a domain, Z is an *analytic variety* if it is the zero set of functions $f_1, \ldots, f_q$ regular on U.

Lastly, we will introduce the structure sheaf $\mathscr{O}_Z$ on the analytic variety Z. If U is open in Z, let $\mathscr{O}_Z = \mathscr{H}_U / I_Z$, where $\mathscr{H}_U$ is the restriction of the sheaf of germs of holomorphic functions onto U, and $I_Z = (f_1, \ldots, f_q)$ for some $q \in \mathbf{Z}^+$. Then

Definition 4.1.2. An *analytic space* $(X, \mathscr{O}_X)$ is a ringed space such that around every point $x \in X$, there exists an open neighborhood $U \subset X$ such that $(U, \mathscr{O}_U)$ is isomorphic to an analytic variety $(Z, \mathscr{O}_Z)$.

If $U \subset \mathbf{C}^n$, then we say that U is *analytic* if, for each $x \in U$, there are functions $f_1, \ldots, f_q$, holomorphic in a neighborhood V of x, such that $V \cap U$ is identical to the set of points $x \in V$ satisfying the equations $f_i(z) = 0$ for $i = 1, \ldots, q$.

An interesting revelation of Chow's theorem is that while it is not surprising that algebraic subspaces could be analytic, on the other hand, analytic subspaces of a projective space are indeed algebraic. To illustrate our point consider the following example.

Certainly, the set where $\sin x$ vanishes in $\mathbb{A}^1$ is analytic. However, one observes that if V is a set of zeroes of a finite number of polynomials (that are not all zero), V is finite. Hence, algebraic sets are either finite or the whole parent space. Going back to $\sin x$, its set of roots is infinite and as a result, this analytic set is not algebraic! Granted, $\mathbf{A}$ is not compact or projective and Chow's theorem still holds.

4.2 Analytic Sheaves

We call a sheaf that is a sheaf of $\mathscr{H}$-modules an analytic sheaf.

Let $X \subset \mathscr{C}^n$. If $\mathscr{C}(X)_x$ denotes the sheaf of germs of continuous functions defined on X defined at x, then the sheaf $\mathscr{H}_{x,X}$ of germs of holomorphic functions on X at x is a subsheaf of $\mathscr{C}(X)_x$. Now, if

$$\epsilon_x : \mathscr{C}(\mathbf{C}^n)_x \longrightarrow \mathscr{C}(X)_x$$

is the restriction of the germs of functions in $\mathbf{C}^n$ to X, then clearly $\epsilon_x[\mathscr{H}_x] = \mathscr{H}_{x,X}$, where $\mathscr{H}$ is the holomorphic functions defined on $\mathbf{C}^n$. Moreover, if $\mathscr{A}_x(X)$ denotes $\ker \epsilon_x$, we can identify $\mathscr{H}_{x,X}$ as the quotient $\mathscr{H}_x/\mathscr{A}_x(X)$. Lastly, as $\mathscr{H}_X$ is the union of $\mathscr{H}_{x,X}$ over $x \in X$, we would like $\mathscr{H}_X$ to be identical to the restriction of $\mathscr{H}/\mathscr{A}(X)$ to X.

Now, suppose Y is a closed analytic subset of X, then under a similar construction as above, we identify $\mathscr{H}_{x,Y}$ as $\mathscr{H}_{x,X}/\mathscr{A}_{x,Yz}$, which is a stalk of $\mathscr{H}_X/\mathscr{A}_x(Y)$. But since Y closed, $\mathscr{H}_X/\mathscr{A}_x(Y)$ is zero outside of Y, i.e., it is concentrated on Y; hence, its restriction to Y is $\mathscr{H}_Y$.

4.3 Coherent Analytic Sheaves

An analytic sheaf is a sheaf of modules over the sheaf of rings $\mathscr{H}_X$; that is,it is a sheaf of $\mathscr{H}_X$-modules.

Proposition 4.3.1. —

1. *The sheaf $\mathscr{H}_X$ is a coherent sheaf of rings.*
2. *If Y is a closed analytic subspace of X, the sheaf $\mathscr{A}(Y)$ is a coherent analytic sheaf.*

Proof. no.3, prop. 1 [Ser56] □

5 Analytification

In the rest of this section, we will construct a coherent analytic sheaf from a coherent algebraic sheaf on any scheme X. In the next chapter we will focus on just Z-closed subvarieties of the projective space $\mathbb{P}^n_{\mathbf{C}}$

By analytification, we mean two things. One, the process of assigning an analytic space structure to a scheme of finite type over **C**; on the other hand, we also mean associating an analytic sheaf to any (algebraic) sheaf on an a variety. Both processes are functorial.

5.1 Associating an analytic sheaf to a sheaf on a variety

In this section, we will be working with the equivalent definition of sheaves due to Serre [Ser55] Let X be an algebraic variety, and $\mathscr{F}$ is any sheaf on X.

Now X^h denotes the analytic space associated with X. Let $\mathscr{F}$ be any sheaf on X. We want to make it into a sheaf on X^h. Further, we remind the reader $\pi : \mathscr{F} \to X$ is a projection.

To make $\mathscr{F}$ into a sheaf on X^h, we will assign to it an appropriate topology[3]. To do so, consider the injection of $\mathscr{F}$ into $X^h \times \mathscr{F}$ by sending $f \in \mathscr{F}$ to $(\pi(f), f)$. That is, $\mathscr{F}$ inherits the product topology. We denote this new sheaf on sp($\mathscr{F}$) by $\mathscr{F}'$. This in fact makes $\mathscr{F}$ into a structure sheaf on X^h as for any $x \in X$, the stalk $\mathscr{F}'_x$ sits inside the product as $(x, \mathscr{F}_x)$ which we can identify as $\mathscr{F}_x$.

Remark 5.1.1. The sheaf $\mathscr{F}'$ is the inverse image of $\mathscr{F}$ under the continuous map $X^h \to X$.

Definition 5.1.2. Let $\mathscr{F}$ be an algebraic sheaf on X. Then *the analytic sheaf associated to* $\mathscr{F}$ is defined by

$$\mathscr{F}^h = \mathscr{F}' \otimes_{\mathscr{O}_X} \mathscr{H},$$

where the tensor is taken over the sheaf of rings and $\mathscr{H}$ is the sheaf of germs of holomorphic functions defined on X.

As the tensor results a change in the ring of operators of $\mathscr{F}'$ to $\mathscr{H}$, the new sheaf $\mathscr{F}^h$ is a sheaf of $\mathscr{H}$-modules.

Proposition 5.1.3. *The functor $\mathscr{F}^h$ is exact and for every algebraic sheaf $\mathscr{F}$, the homomorphism $\alpha : \mathscr{F}' \to \mathscr{F}^h$ is injective.*

Proof. no.9, prop. 10 [Ser56]. ◻

[3]For now, by $\mathscr{F}$ we mean sp($\mathscr{F}$), that is F as a set, unless otherwise stated.

5.2 Analytic space associated to a scheme

Given a scheme X, we will attach to it an analytic structure. We will denote the resulting analytic space by X^h. Cover X with affine subsets Spec (A_i) where each A_i is an algebra of finite type over $\mathbf{C}$. That is, each

$$A_i \cong \mathbf{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_q)$$

for some polynomials $f_1, \ldots, f_q$.

Since each polynomial $f_1, \ldots, f_q$ is holomorphic their set of zeros is a complex analytic subspace $(Y_i)^h \subset \mathbf{C}^n$. As we glue Spec (A_i) to get the scheme X, with the same gluing data, gluing the Y_i gives an analytic space X^h.

6 GAGA & Chow's Theorem

6.1 GAGA

Theorem 6.1.1. *In full generality, if X is a projective variety, then the following hold:*

1. *For every coherent algebraic sheaf $\mathscr{F}$ on X, and for every integer $q \geq 0$, the natural maps*
$$\epsilon : H^q(X, \mathscr{F}) \to H^q(X^h, \mathscr{F}^h)$$
are isomorphisms.

2. *If $\mathscr{F}$ and $\mathscr{G}$ are two coherent algebraic sheaves on X, every analytic homomorphism of $\mathscr{F}^h$ into $\mathscr{G}^h$ comes from a unique algebraic homomorphism of $\mathscr{F}$ into $\mathscr{G}$.*

3. *For every coherent analytic sheaf $\mathscr{M}$ on X^h, there exists a coherent algebraic sheaf $\mathscr{F}$ on X such that $\mathscr{F}^h$ is isomorphic to $\mathscr{M}$. Moreover, this property determines $\mathscr{F}$ in a unique fashion, up to isomorphism.*

6.2 Chow's Theorem

Corollary 6.2.1 (Chow)**.** *Let $\mathbb{P}$ be a scheme of finite type over* $\mathbf{C}$. *Assume $\mathbb{P}$ is compact. Then any closed analytic subspace of $\mathbb{P}$ is algebraic.*

Proof. Let X be a projective space, and let Y be a closed analytic subset of X^h. Suppose $i : Y \to X^h$ is the inclusion map. By Cartan's theorem [Ser56, NO. 3, PROP 1], $\mathscr{H}_Y = \mathscr{H}_X / \mathscr{A}(Y)$ is a coherent analytic sheaf on X^h. By GAGA, there exists a coherent algebraic sheaf $\mathscr{F}$ on X such that $\mathscr{H}_Y = \mathscr{F}^h$. Since the homomorphism $\alpha : \mathscr{F}' \longrightarrow \mathscr{F}^h$ is injective, the support of $\mathscr{F}^h$ equals the support of $\mathscr{F}$. Since $\mathscr{F}$ is coherent, the support is Z-closed by Corollary 2.3.13. But $\mathscr{F}^h = \mathscr{H}_Y$, so Y is Z-closed as required. □

6.3 Consequences of Chow's Theorem

Chow's theorem affirms the existence of nonconstant meromorphic functions on a compact Riemann surface. Embed the surface into a complex n-dimensional projective space $\mathbb{P}^n$ as a complex submanifold for a suitable n. Then the surface acquires the structure of a smooth projective algebraic curve [Bal10]. Hence, embedded Riemann surfaces as well us meromorphic functions on them can also be described algebraically.

Corollary 6.3.1. *Every compact Riemann surface is an algebraic curve.*

That is, as any analytic subset of $\mathbb{P}^n$ is an algebraic subset, any complex submanifold of $\mathbb{P}^n$ is an algebraic submanifold [Kod12]. On the other hand, we have that any algebraic curve is a compact Riemann surface—an algebraic manifold of dimension 1 (since we are working over $\mathbb{C}$).

Another consequence is the following:

Corollary 6.3.2 ([Ara65]:p.264)**.** *A holomorphic map between nonsingular projective algebraic varieties is a morphism of varieties.*

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