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1 Abstract

Recent investigations of various quantum-gravity theories have revealed a variety of possible mechanisms that lead to Lorentz violation. One of the more elegant of these mechanisms is known as Spontaneous Lorentz Symmetry Breaking (SLSB), where a vector or tensor field acquires a nonzero vacuum expectation value. As a consequence of this symmetry breaking, massless Nambu-Goldstone modes appear with properties similar to the photon in Electromagnetism. This thesis considers the most general class of vector field theories that exhibit spontaneous Lorentz violation–known as bumblebee models–and examines their candidacy as potential alternative explanations of E&M, offering the possibility that Einstein-Maxwell theory could emerge as a result of SLSB rather than of local U(1) gauge invariance. With this aim we employ Dirac's Hamiltonian Constraint Analysis procedure to examine the constraint structures and degrees of freedom inherent in three candidate bumblebee models, each with a different potential function, and compare these results to those of Electromagnetism. We find that none of these models share similar constraint structures to that of E&M, and that the number of degrees of freedom for each model exceeds that of Electromagnetism by at least two, pointing to the potential existence of massive modes or propagating ghost modes in the bumblebee theories.

2 Acknowledgements

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Lastly I would like to thank the Department of Physics and Astronomy at Colby College for imparting–or attempting to–all of its wisdom about the most beautiful subject on this earth to an eager, excited, and curious undergraduate. Also, a sincere thank you to the National Science Foundation for the funding that facilitated the research behind this entire project.

Oh, and Nutritious: RLFA. Always.

Contents

3 Introduction

Lorentz symmetry is one of the most important cornerstones of our physical understanding of the universe: it states that, for an observer, the laws of physics are the same irrespective of his or her orientation or constant velocity. In other words, Lorentz symmetry implies that all directions and all uniform velocities are equivalent. This idea became widely accepted after Albert Einstein formulated it as part of his theory of special relativity in 1905, and since then Lorentz symmetry has been tested without failure in numerous experiments. Even more, this symmetry is crucial both in particle physics (the Standard Model) and the theory of general relativity [1].

However, here we find our problem: there is a fundamental discord between the Standard Model and General Relativity. While taken together these two disparate theories are successful in describing all phenomena and experimental results, they cannot be reconciled and unified to form a single, ultimate 'theory of everything.' No widely accepted theory of quantum gravity has yet been created, and physics at the Planck scale ($\approx 10^{-35}$ m for length) where the worlds of quantum physics and general relativity meet remains currently beyond theoretical description.

In an attempt to attack these yet unsolved problems, theorists consider the possibility of Lorentz symmetry breaking at the aforementioned scale. Investigations of quantum-gravity theories have uncovered a variety of possible mechanisms that can lead to Lorentz violation. Mechanisms allowing for such symmetry violations have been found in string theory [2], including the possibility that Lorentz symmetry is spontaneously broken. This type of symmetry breaking is well known and particularly elegant in particle physics.

In general terms, symmetry of a given system breaks spontaneously when a completely symmetric set of conditions or equations results in an asymmetry in the sytem. A cylindrical stick on a rigid surface with a force exerted on the stick vertically down along its symmetry axis forms a simple example of symmetry breaking. While the system is completely symmetric with respect to rotations around the symmetry axis, if a sufficiently large force is applied, the system becomes unstable and the stick bends in some direction breaking the axial symmetry [1]. For an arbitrary tensor field, the process of spontaneous symmetry breaking (SSB) can be very complex. A simpler case, then, is to consider a theory with a vector field that undergoes SSB.

This thesis describes research done in investigating the effects of vector field theories that exhibit spontaneous Lorentz symmetry breaking (SLSB): such a violation occurs when the vector field acquires a nonzero vacuum expectation value. The presence of a background value provides signatures of Lorentz violation that can also be probed experimentally; the theoretical framework for their investigation is given by the Standard Model Extension (SME) [3, 4].

One important consequence of spontaneous symmetry breaking in field theory is what's known as Goldstone's theorem, which states that when a continuous global symmetry is spontaneously broken massless modes–called Nambu-Goldstone modes–appear [5]-[8]. These massless modes remain in the space of solutions that minimize the potential (degenerate solutions); *continuous* here means that the symmetry transformations are continuous as opposed to discrete; *global* instead of local means that the transformations are the same for all spacetime points. If a *local* symmetry is broken, then alternatively the Higg's mechanism can occur where the massless modes give rise instead to massive gauge fields.

Many investigations to date have concentrated on the case of a vector field acquiring a nonzero vacuum value. These theories, called bumblebee models [2, 3, 5], are the simplest examples of field theories with spontaneous Lorentz breaking. Bumblebee models can be defined with different forms of the potential and kinetic terms for the vector field, and with different couplings to matter and gravity [9]-[16]. They can be considered as well in different spacetime geometries, including Riemann, Riemann-Cartan, and in Minkowski spacetimes, the last of which is the case for the analysis in this paper.

Much of the interest in bumblebee models stems from the fact that they are theories without local U(1) gauge symmetry–this is due to the presence of a potential V that breaks the gauge symmetry–but which nonetheless allow for the propagation of massless vector modes. Theories with spontaneous Lorentz violation can also exhibit a variety of physical effects due to the appearance of both Nambu-Goldstone (NG) and massive Higgs modes [5, 7]. Indeed, one idea is that bumblebee models, with appropriate kinetic and potential terms, might provide alternative descriptions of photons besides that given by local U(1) gauge theory. In this scenario, massless photon modes arise as NG modes when Lorentz violation is spontaneously broken. However, in addition to lacking local U(1) gauge invariance, bumblebee models differ from electromagnetism (in flat or curved spacetime) in a number of other ways. For example, the kinetic terms need not have a Maxwell form. Instead, in this paper we consider the generalized vector theories of the form

$$\mathcal{L} = a_1(\partial_\mu A_\nu)(\partial^\mu A^\nu) + a_2(\partial_\mu A^\mu)(\partial_\nu A^\nu) + a_3(\partial_\mu A_\nu)(\partial^\nu A^\mu) - V(A_\mu A^\mu \pm b^2) \tag{1}$$

with arbitrary coefficients a_1, a_2, a_3 that determine the form of the kinetic terms for the bumblebee field, of the style in Will-Nordvedt vector-tensor theories [17, 18]. Such arbitrary coefficients may involve the introduction of ghost modes into the theory. Further differences arise due to the presence of a potential term V in the Lagrangian density for bumblebee models. It can take a variety of forms, which may involve additional excitations due to the presence of massive modes or Lagrange-multiplier fields that have no counterparts in electromagnetism.

The potential $V(A_\mu A^\mu \pm b^2)$ has a minimum with respect to its argument or is constrained to zero when

$$A_\mu A^\mu \pm b^2 = 0. \tag{2}$$

This condition is satisfied when the vector field has a nonzero vacuum value

$$A_\mu = \langle A_\mu \rangle = b_\mu \tag{3}$$

with

$$b_\mu b^\mu = \mp b^2. \tag{4}$$

Here the upper sign is for a spacelike vector and the lower sign is for a timelike vector. It is *this* vacuum value that spontaneously breaks local Lorentz invariance.

There are many forms that can be considered for the potential $V(A_\mu A^\mu \pm b^2)$. These include functionals involving Lagrange-multiplier fields, as well as both polynomial and nonpolynomial functionals in $(A_\mu A^\mu \pm b^2)$ [1, 9]. For simplicity, three limiting-case examples are considered here. The first introduces a Lagrange-multiplier field λ and has a linear form,

$$V = \lambda(A_\mu A^\mu \pm b^2). \tag{5}$$

The second is a smooth quadratic potential

$$V = \frac{1}{2}\kappa(A_\mu A^\mu \pm b^2)^2, \tag{6}$$

where κ is a constant. The third again involves a Lagrange-multiplier field λ, but has a quadratic form,

$$V = \frac{1}{2}\lambda(A_\mu A^\mu \pm b^2)^2. \tag{7}$$

Models with potentials (5) and (6) inducing spontaneous symmetry breaking were investigated by Kostelecký and Samuel [1], while the potential (7) was recently examined in [6]. Models with a linear Lagrange-multiplier potential (5) but arbitrary coefficients a_1, a_2, a_3 are special cases (with a fourth-order term in A_μ omitted) of the models described in [10].

It has been shown that, with particular choices of coefficients, bumblebee theories with the various potentials shown above resemble electromagnetism in many respects. The goal of this paper is to investigate further the extent to which these bumblebee models can be considered as equivalent to electromagnetism; this question is examined here in flat spacetime. In such a Minkowski spacetime, the main differences between bumblebee models and electromagnetism are due to the nature of the constraints imposed on the field variables and in the number of physical degrees of freedom permitted by the theory. The determination of physical degrees of freedom inherent in our bumblebee models is of particular interest, since extra degrees of freedom (with respect to those of electromagnetism) could potentially provide insight into the origins or behaviors of dark matter and/or dark energy, or perhaps point to modifications of gravity theory itself. However, these extra degrees of freedom may also be signatures of unphysical behavior or the presence of ghost modes in our bumblebee models. Thus, to explicitly investigate these concerns, a Hamiltonian constraint analysis [19]-[22] is used.

The reader is assumed to have some familiarity with tensor notation and the use of a metric tensor to lower and raise indices. An introduction to this topic is provided in [23]. The form of the metric tensor,

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{8}$$

is used throughout this thesis as the metric tensor in flat spacetime. Two equivalent notations for a derivative are used, illustrated as follows:

$$\frac{\partial \phi}{\partial x^\mu} = \partial_\mu \phi \tag{9}$$

Repeated indices imply a summation; Triple differentials of form d^3x are written simply in short as dx. Finally, all work is done in units where the speed of light in vacuum c, electric constant ϵ_0, and $\hbar$ are all set equal to one.

4 Hamiltonian Constraint Analysis

4.1 Origins

The Hamiltonian constraint analysis was first introduced by P. A. M. Dirac in [24] in an attempt to generalize and expand methods used in *quantization* – constructing quantum theories of systems from their corresponding classical theories. In the canonical quantization approach one first must utilize the Hamiltonian formulation of the classical mechanics of the given system. It is done by constructing a Hamiltonian in terms of generalized coordinates and momenta and then using Hamilton's equations to derive the equations of motion for the system. All physical quantities are expressed in terms of these generalized coordinates and momenta. Quantization then is achieved by switching from physical quantities to operators and by imposing canonical commutation relations between momentum and coordinate operators. The evolution of the system is required to obey Schrödinger equation [21].

When applying this approach to field theories, many of which are formulated as Lagrangians, the necessary starting point is often a Lagrangian formulation which is then transformed to a Hamiltonian formulation. Since a Lagrangian is expressed in terms of generalized coordinates and velocities, while a Hamiltonian employs generalized coordinates and momenta, this requires reexpressing velocities in terms of coordinates and momenta. However, in a class of theories, called singular theories, generalized momenta are not necessarily independent functions of generalized velocities. Velocities then cannot be expressed uniquely in terms of momenta and a different approach is needed [24].

Dirac was first to develop this approach. Even more, it was found that a singularity of a theory implies that there exist essential relations of the form

$$\phi(q,p) = 0, \tag{10}$$

called constraints, between coordinates q and momenta p, hence the name for the method. Dirac's constraint analysis is particularly useful in identifying the physical degrees of freedom in a theory; each identified 'constraint' removes a degree of freedom from the theory.

4.2 Overview

The Hamiltonian Constraint Analysis method starts with a Lagrangian density $\mathcal{L}$ for a given theory, which essentially encodes all of the information concerning its dynamics. We use this to find the generalized momenta according to

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)}, \tag{11}$$

where A_μ is the dynamic field. At this point, any relations of form

$$\phi_m(A_\mu, \Pi^\nu) \approx 0, \;\; m \in [1, ..., M] \tag{12}$$

are immediately identified as *primary constraints*, where m is an index. We will assume that a total of M primary constraints has been identified. Here, Dirac's weak equality symbol '$\approx$' is used to denote equations that hold as a result of imposing a constraint. We then form the Hamiltonian Density,$\mathcal{H}$, which encodes information concerning the *energy* of the system, as follows:

$$\mathcal{H} = \Pi^\mu \partial_0 A_\mu - \mathcal{L}. \tag{13}$$

Using the momenta expressions obtained before in (11), we rewrite the Hamiltonian density as a function of *only* fields and momenta, with no time derivatives of either. This corresponds to switching from a Lagrangian expressed in terms of generalized coordinates and velocities to a Hamiltonian expressed, instead, solely in coordinates and momenta. We then calculate the Hamiltonian from the Hamiltonian Density according to

$$H = \int dx\, \mathcal{H}(x). \tag{14}$$

The Hamiltonian is used in Hamilton's equations to describe the dynamics of the system. These equations can be written in short using Poisson's bracket notation, defined for two functions $f = f(A_\mu, \Pi^\mu)$ and $g = g(A_\mu, \Pi^\mu)$ as

$$[f(x), g(y)] = \int dz \left(\frac{\partial f(x)}{\partial A_\mu(z)} \frac{\partial g(y)}{\partial \Pi^\mu(z)} - \frac{\partial f(x)}{\partial \Pi^\mu(z)} \frac{\partial g(y)}{\partial A_\mu(z)} \right). \tag{15}$$

Using this notation, Hamilton's equations of motion for any quantity $f = f(A_\mu, \Pi^\mu)$ can be written as

$$\dot{f} = [f, H] + \frac{\partial f}{\partial t}, \tag{16}$$

where the second term accounts for quantities with explicit time dependence. Now, the Hamiltonian defined above in (14) is not uniquely determined, since we may add to it any linear combination of the primary constraints (12), which are zero. We can thus construct what's called the Total Hamiltonian, H_T:

$$H_T = \int dx\ \mathcal{H}(x) + \int dx\ u_m \phi_m,\ \left(m \in [1, ..., M]\right), \tag{17}$$

where we have added the primary constraints to the Hamiltonian in (14) multiplied by corresponding coefficients u_m, which can be functions of both fields and momenta. In fact, it can be shown that a solution of the form

$$u_m = U_m(A_\mu, \Pi^\nu) \tag{18}$$

MUST exist if the Lagrangian equations of motion for the system are consistent [24]. The addition of these constraints suggests that the Hamiltonian is not uniquely determined away from the constraint surface, and thus our theory cannot dynamically distinguish H_T from H. Now, this Total Hamiltonian is used to explore the time evolution of the primary constraints according to

$$\dot{\phi_m} = [\phi_m, H_T] + \frac{\partial \phi_m}{\partial t}. \tag{19}$$

In this method the constraints are required to be valid at all times and equal to zero, thus

$$\dot{\phi_m} \approx 0. \tag{20}$$

Equations (19) and (20) are used in conjunction to produce one of the following five cases:

- **Case 1: Inconsistencies** $\Rightarrow$ Mathematically inconsistent equalities of the type $1 = 0$. This case is indicative of a Lagrangian that gives rise to inconsistent equations of motion, and should thus be discarded.

- **Case 2: Trivial Solutions** $\Rightarrow$ Mathematical equalities that reduce to $0 = 0$. Equations such as these are automatically satisfied (often with the help of the primary constraints ϕ_m) and require no further investigation.

- **Case 3: Equations Involving u_m Coefficients**$\Rightarrow$ Equations of this type will all eventually be used in conjunction to solve for the u_m's in terms of the fields and conjugate momenta of the theory, $u_m(A_\mu, \Pi^\nu)$.

- **Case 4: Equations/Relations NOT Involving Fields or Momenta** $\Rightarrow$ These are simply additional equations that must be satisfied for a consistent theory.

- **Case 5: Equations Involving Just Fields and/or Momenta** $\Rightarrow$ Such an equation is necessarily independent of the primary constraints ϕ_m, and borrowing from Dirac's notation, it is thus of the form $\chi(A_\mu, \Pi^\nu) \approx 0$. These imply more constraints on the Hamiltonian variables, and thus equations that turn up in this fashion are called *Secondary Constraints.*

Expounding upon **Case 5** above, we use the following notation for Secondary Constraints:

$$\phi_n(A_\mu, \Pi^\nu) \approx 0,\ n \in [1, ..., N] \tag{21}$$

where n is their appropriate index. We will assume that a total of N secondary constraints are identified in the manner above. The next step is to then calculate the time evolution of these Secondary Constraints, using the Total Hamiltonian above:

$$\dot{\phi_n} = [\phi_n, H_T] + \frac{\partial \phi_n}{\partial t} \approx 0. \tag{22}$$

Equation (22) again results in one of the five cases above. **Cases 1-5** should then again serve for classification, using **Case 5** almost as a 'feedback loop' until all such equations fall into **Cases 2-4**, which they should for a coherent field theory. At this point all Primary Constraints ϕ_m and all Secondary Constraints ϕ_n have been identified, all (if any) relations for the u_m coefficients have been discovered, and any additional consistency equations have also been exposed.

We can now move to the classification of the constraints, by examining

$$[\phi_m, \phi_{m'}]\,, [\phi_m, \phi_n]\,, [\phi_n, \phi_{n'}]\,,\ m \neq m',\ n \neq n'. \tag{23}$$

Any constraint subject to the calculations in (23) falls into one of two categories:

- **Category A** $\Rightarrow$ Its Poisson Bracket with *all* other constraints is zero; it commutes with all other constraints.

- **Category B** $\Rightarrow$ Its Poisson Bracket with *any* other constraint is non-zero; it does not commute with all other constraints.

Constraints that fall into **Category A** are hereafter referred to as *First-Class Constraints*

$$\phi_k(A_\mu, \Pi^\nu) \approx 0, \ k \in [1, ..., K] \tag{24}$$

where k is their appropriate index. We will assume that a total of K First-Class constraints has been classified in this manner. Constraints that fall into **Category B** are hereafter referred to as *Second-Class Constraints*

$$\phi_j(A_\mu, \Pi^\nu) \approx 0, \ j \in [1, ..., J = ((M + N) - K)] \tag{25}$$

where j is the appropriate index for these constraints; We assume a total of J Second-Class constraints have been classified. At this point, we have classified all of the constraints, and we can now use them to construct the Extended Hamiltonian, H_E. Dirac showed that a solution of the equation in (18) is not unique, and if one exists that it can be added to any solution of the homogeneous equations associated with (19) and (22) which is of the form in (24). Since we want the most general solutions to these equations, in the Extended Hamiltonian we add these first-class constraints multiplied by completely arbitrary coefficients v_k:

$$H_E = \int dx \ \mathcal{H}(x) + \int dx \ u_m \phi_m + \int dx \ v_k \phi_k, \ m \in [1, ..., M], \ k \in [1, ..., K]. \tag{26}$$

Again, it is important to take a break here for some clarification. The Extended Hamiltonian above contains *both* Primary *and* First-Class Constraints, each multiplied by their corresponding coefficients, u_m and v_k respectively. However, the v_k coefficients of the First-Class Constraints will remain undetermined throughout this analysis procedure, and can be completely arbitrary functions. This is in direct contrast to the u_m coefficients, which may or may not have had solutions after **Cases 1-5**, but in any case are not arbitrary and must satisfy consistency conditions. However, we are free to take the v_k's to be arbitrary functions of time and we have still satisfied all of the requirements of our dynamical theory. It is entirely possible that a Primary Constraint ϕ_m ultimately be classified as a First-Class Constraint as well, in which case its u_m is completely arbitrary, and cannot and will not be determined; for such a constraint, effectively

$$\phi_m \longrightarrow \phi_k \Rightarrow u_m \longrightarrow v_k. \tag{27}$$

Still, each constraint can appear in (26) *only once*. It's worth it to explicitly show all four possible Constraint classifications:

- **Primary, First-Class** $\Rightarrow$ In this case, the constraint is present in the Total Hamiltonian as ϕ_m with its associated coefficient u_m, but after classification $u_m \longrightarrow v_k$ and $\phi_m \longrightarrow \phi_k$, and it exists in the Extended Hamiltonian only as ϕ_k.

- **Primary, Second-Class** $\Rightarrow$ The constraint remains ϕ_m, and is present in the Extended Hamiltonian only as such, with its associated coefficient u_m.

- **Secondary, First-Class** $\Rightarrow$ In this case the original constraint ϕ_n isn't present in the Total Hamiltonian, but after classification $\phi_n \longrightarrow \phi_k$, where it exists in the Extended Hamiltonian only as such, with its associated coefficient v_k.

- **Secondary, Second-Class** $\Rightarrow$ In this case, the constraint isn't present in either the Extended or Total Hamiltonian.

Now that we have covered all possible Constraint classifications, we employ the Extended Hamiltonian in (26) to calculate the Field and Momenta equations of motion according to

$$\dot{f} = [f, H_E] + \frac{\partial f}{\partial t}. \tag{28}$$

Finally, with all of the constraints identified and classified and all of the Field and Momenta equations of motion obtained, we can begin to make some statements about the degrees of freedom of the particular theory we have analyzed. Each identified constraint effectively removes a degree of freedom from a theory. Additionally, the existence of a First-Class Constraint results in the addition of an arbitrary function v_k to the Extended Hamiltonian H_E, which is thus also present in the theory's equations of motion. Dirac showed that Primary First-Class Constraints are associated with additional unphysical or gauge degrees of freedom, and conjectured the same is true for Secondary First-Class Constraints. From this, a counting rule can be established: for a theory with N total Field and Conjugate Momenta components, if there are n_1 First-Class Constraints and n_2 Second Class Constraints, there will generally be a total of $(N - 2n_1 - n_2)$ independent physical degrees of freedom.

4.3 Regularity vs. Irregularity

Dirac's analytic method is very successful in finding the gauge symmetries present in a theory, and also in identifying and classifying the constraints, which are local functions of the phase space coordinates q and p. The constraints are required to be preserved in the mathematical procedure for the evolution of successive constraints for the sake of procedural consistency; if these constraints are not functionally independent, then it can be shown that Dirac's procedure is entirely unapplicable [25]. There are a series of *regularity conditions* that test the functional independence of the constraints; systems and constraints that fail such a test are called *irregular*. Thus the constraints discovered using Dirac's method essentially fall into two categories–*regular* and *irregular*–and a brief explanation of both is given below.

4.3.1 Regular Constraints

The tests for regularity are fairly thorough, and the reader interested in the specific mathematical details underlying the procedure is referred to [25]. Essentially they rest on the behavior resulting from small variations of the imposed constraint evaluated on the constraint surface, and whether or not this behavior is linearly independent of variations in the phase-space variables themselves. Mathematically it suffices to show that for a set of nonlinear constraints, if and only if the Jacobian, composed of variations in its constraints with respect to variations in the phase-space variables and evaluated on the constraint surface, is of maximal rank, then the set of constraints is regular.

4.3.2 Irregular Constraints

All constraints that fail the test above are said to be *irregular*, and we can classify them further based on their approximate behavior near the constraint surface.

Linear Irregular Constraints are essentially regular systems in disguise, with the mathematical requirements for regularity failing simply due to redundancies in the constraint structure.

Higher-Order Irregular Constraints are of the form

$$\phi \equiv [f(A_a, \Pi^a)]^k \approx 0, \; (k > 1). \tag{29}$$

Constraints of this form are often referred to as *nonlinear constraints* and they are a very interesting and curious aspect of this analytic method, particularly because it isn't yet completely determined how to deal with them in the full context of the Hamiltonian Constraint approach [22, 25]. They do indeed possess linear, regular approximations,

$$\chi \equiv f(A_a, \Pi^a) \approx 0, \tag{30}$$

in the vicinity of the constraint surface, but a naive substitution of such equivalent regular constraints has been proven to occasionally alter the dynamics of the system. In this manner, treatment of such nonlinear, irregular constraints is indeed a sensitive issue; it has been shown that the issue boils down to whether or not the linearized, regular substitution for the irregular constraint is first- or second-class. Namely it makes a difference whether the χ in question can generate a transformation in phase space that leaves the Hamiltonian action unchanged or not.

If the χ is second-class, then it is geometrically equivalent to its linear substitution, and it not only defines the same constraint surface but also yields equivalent dynamical descriptions to a Lagrangian approach. In cases such as these we can essentially replace the irregular, nonlinear constraint with its regular, linear counterpart

$$\phi \equiv [f(A_a, \Pi^a)]^k \approx 0 \;\Rightarrow\; \chi \equiv f(A_a, \Pi^a) \approx 0, \; (k > 1), \tag{31}$$

which is to say that in this case these can be considered on equal footing in the context of Poisson Bracket calculations, Hamilton's equations of motion, and the time-evolution of constraints.

However, if the χ proves to be first-class, then we are forced to consider

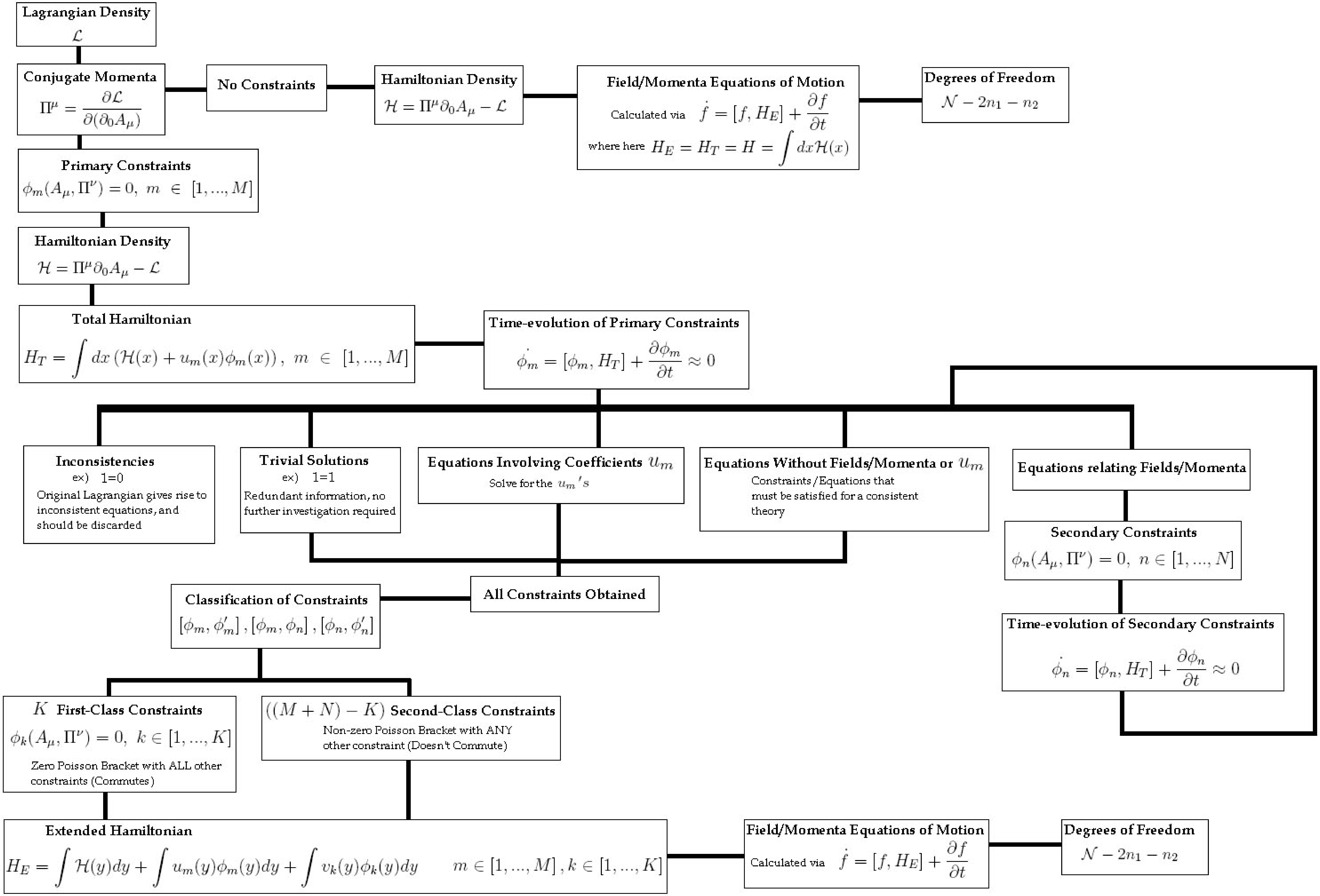
$$\phi \equiv [f(A_a, \Pi^a)]^k \approx 0 \; (k > 1) \tag{32}$$

and

$$\chi \equiv f(A_a, \Pi^a) \approx 0 \tag{33}$$

as *completely distinct*, and (31) does not hold. This distinction must be preserved in Poisson Bracket calculations, Hamiltonian's Equations of motion, and the time-evolution of constraints; in these cases the substitution generates a system whose dynamics are different from those obtained via the Euler-Lagrange equations.

The Hamiltonian Constraint Analysis procedure is admittedly a very intricate and detailed one, involving a lot of esoteric terminology and mathematical tools. For this reason, it may help to consult the flow chart on the following page for visual assistance with the scope and sequential nature of this process. For a more thorough theoretical and mathematical understanding, the reader is directed to [19]-[22], and [26].


Lagrangian Density
$\mathcal{L}$
Conjugate Momenta
$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)}$
No Constraints
Hamiltonian Density
$\mathcal{H} = \Pi^\mu \partial_0 A_\mu - \mathcal{L}$
Field/Momenta Equations of Motion
Calculated via $\dot{f} = [f, H_E] + \frac{\partial f}{\partial t}$
where here $H_E = H_T = H = \int dx \mathcal{H}(x)$
Degrees of Freedom
$\mathcal{N} - 2n_1 - n_2$
Primary Constraints
$\phi_m(A_\mu, \Pi^\nu) = 0,\ m \in [1, ..., M]$
Hamiltonian Density
$\mathcal{H} = \Pi^\mu \partial_0 A_\mu - \mathcal{L}$
Total Hamiltonian
$H_T = \int dx\, (\mathcal{H}(x) + u_m(x)\phi_m(x)),\ m \in [1, ..., M]$
Time-evolution of Primary Constraints
$\dot{\phi_m} = [\phi_m, H_T] + \frac{\partial \phi_m}{\partial t} \approx 0$
Inconsistencies
ex) 1=0
Original Lagrangian gives rise to inconsistent equations, and should be discarded
Trivial Solutions
ex) 1=1
Redundant information, no further investigation required
Equations Involving Coefficients u_m
Solve for the $u_m{}'s$
Equations Without Fields/Momenta or u_m
Constraints / Equations that must be satisfied for a consistent theory
Equations relating Fields/Momenta
Secondary Constraints
$\phi_n(A_\mu, \Pi^\nu) = 0,\ n \in [1, ..., N]$
Time-evolution of Secondary Constraints
$\dot{\phi_n} = [\phi_n, H_T] + \frac{\partial \phi_n}{\partial t} \approx 0$
All Constraints Obtained
Classification of Constraints
$[\phi_m, \phi'_m], [\phi_m, \phi_n], [\phi_n, \phi'_n]$
K First-Class Constraints
$\phi_k(A_\mu, \Pi^\nu) = 0,\ k \in [1, ..., K]$
Zero Poisson Bracket with ALL other constraints (Commutes)
$((M + N) - K)$ Second-Class Constraints
Non-zero Poisson Bracket with ANY other constraint (Doesn't Commute)
Extended Hamiltonian
$H_E = \int \mathcal{H}(y)dy + \int u_m(y)\phi_m(y)dy + \int v_k(y)\phi_k(y)dy \quad m \in [1, ..., M], k \in [1, ..., K]$
Field/Momenta Equations of Motion
Calculated via $\dot{f} = [f, H_E] + \frac{\partial f}{\partial t}$
Degrees of Freedom
$\mathcal{N} - 2n_1 - n_2$

5 Application to Vector Theories

5.1 Electricity and Magnetism

5.1.1 Overview

We begin by subjecting the Electromagnetic Field Theory to the Hamiltonian Constraint Analysis discussed above. In addition to illuminating information concerning the constraint structure, degrees of freedom, and Hamilton's equations of motion for the theory, we will use the results of this particular analysis as a basis of comparison for our candidate Bumblebee models. This example was analyzed in [26]. Here, a brief summary is presented to provide the framework within which we will attempt to compare the results of our Bumblebee models.

5.1.2 Analysis

We start with the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\rho J^\rho, \tag{34}$$

where $F^{\mu\nu}$ is the Electromagnetic Field Tensor, defined as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix} \tag{35}$$

and where the charge density ρ and the current density $\vec{J}$ are joined in a current four-vector

$$J^\mu = \left(\rho, \vec{J}\right). \tag{36}$$

We can expand the Lagrangian above to

$$\mathcal{L} = \frac{1}{2}(\partial_0 A_i)^2 + \frac{1}{2}(\partial_i A_0)^2 - \frac{1}{2}(\partial_k A_l)^2 + \frac{1}{2}(\partial_k A_l)(\partial_l A_k) - (\partial_i A_0)(\partial_0 A_i) - A_\rho J^\rho. \tag{37}$$

First, the conjugate momenta are found using Eq.(11):

$$\Pi^0 = 0 \tag{38}$$

$$\Pi^i = \partial_0 A_i - \partial_i A_0 \tag{39}$$

Above, we recognize that Eq.(38) defines a primary constraint:

$$\phi_1 = \Pi^0. \tag{40}$$

We calculate the Hamiltonian density according to Eq.(13) and get that

$$\mathcal{H} = \frac{1}{2}(\Pi^i)^2 + \Pi^i\partial_i A_0 + \frac{1}{2}(\partial_i A_j)^2 - \frac{1}{2}(\partial_j A_i)(\partial_i A_j) + A_\mu J^\mu \tag{41}$$

We proceed by investigating the time evolution of the primary constraint ϕ_1 according to Eq.(19):

$$\dot{\phi}_1 = [\phi_1, H] + \frac{\partial \phi_1}{\partial t}. \tag{42}$$

From (15),

$$[\phi_1, H] = \int dz \left(\frac{\partial \phi_1(x)}{\partial A_\mu(z)} \frac{\partial H(y)}{\partial \Pi^\mu(z)} - \frac{\partial \phi_1(x)}{\partial \Pi^\mu(z)} \frac{\partial H(y)}{\partial A_\mu(z)} \right) \tag{43}$$

Clearly, from Eq.(40),

$$\frac{\partial \phi_1(x)}{\partial A_\mu(z)} = 0 \tag{44}$$

and

$$\frac{\partial \phi_1(x)}{\partial t} = 0. \tag{45}$$

However,

$$\frac{\partial \phi_1(x)}{\partial \Pi^\mu(z)} = \frac{\partial \Pi^0(x)}{\partial \Pi^\mu(z)} = \delta^0_\mu \delta(z-x) \tag{46}$$

and

$$\frac{\partial H(y)}{\partial A_\mu(z)} = \delta^0_\mu \left(-\frac{\partial \Pi^i(z)}{\partial z^i} \right) + \delta^j_\mu \left(-\frac{\partial}{\partial z^i} \frac{\partial A_j(z)}{\partial z^i} + \frac{\partial}{\partial z^i} \frac{\partial A_i(z)}{\partial z^j} \right) + \delta^\mu_\nu \left(J^\nu(z) \right) \tag{47}$$

Thus,

$$\dot{\phi}_1 = \frac{\partial \Pi^i(x)}{\partial x^i} - J^0(x) \tag{48}$$

Requiring that the time evolution of Primary Constraints is constant, we obtain that

$$\frac{\partial \Pi^i(x)}{\partial x^i} - J^0(x) = 0 \tag{49}$$

Eq.(49) defines a Secondary Constraint ϕ_2:

$$\phi_2 = \frac{\partial \Pi^i(x)}{\partial x^i} - J^0(x) \tag{50}$$

We continue by investigating the time evolution of the Secondary Constraint ϕ_2. In this case, we use the total Hamiltonian as dictated and defined by Eq.(17), obtaining

$$\dot{\phi}_2 = [\phi_2, H_T] + \frac{\partial \phi_2}{\partial t} \tag{51}$$

and hence

$$[\phi_2(x), H_T(y)] = \left[\phi_2(x), \int dy\, \mathcal{H}(y) \right] + \left[\phi_2(x), \int dy\, u_1(y)\phi_1(y) \right] \tag{52}$$

First, we find that

$$\left[\phi_2(x), \int dy\, \mathcal{H}(y) \right] = -\frac{\partial J^j}{\partial x^j}. \tag{53}$$

Second, we find that

$$\left[\phi_2(x), \int dy\, u_1(y)\, \phi_1(y)\right] = 0. \tag{54}$$

In addition,

$$\frac{\partial \phi_2}{\partial t} = -\frac{\partial J^0}{\partial t} \tag{55}$$

Thus,

$$\dot{\phi}_2 = -\frac{\partial J^j}{\partial x^j} - \frac{\partial J^0}{\partial t} = -\frac{\partial J^\mu}{\partial x^\mu} = 0, \tag{56}$$

where we have already required that the constraint remains constant with time. It's important to note that Eq.(56) is an example of **Case 4** as discussed in the section above, and does not contain field or momentum variables. As such it does not breed any further constraints, yet it is an expression which must remain valid at all times for the given theory to be consistent. We have thus shown that the constraints truncate with ϕ_2, and can proceed to classify them. It is not hard to show that

$$[\phi_1, \phi_2] = 0, \tag{57}$$

thus

$$(\phi_1, \phi_2) = \mathbf{First-Class\ Constraints}. \tag{58}$$

According to Eq.(26), the extended Hamiltonian H_E then is:

$$H_E(x) = \int dx\, \mathcal{H}(x) + \int dx\, v_1(x)\, \phi_1(x) + \int dx\, v_2(x)\phi_2(x), \tag{59}$$

where v_1 and v_2 are arbitrary functions, as discussed previously. Following the general form of Eq.(28), we can calculate the equations of motion for the fields A_μ and corresponding momenta Π^μ. The results are simply given below:

$$\dot{A}_0 = v_1(x) \tag{60}$$

$$\dot{A}_i = \Pi^i(x) + \frac{\partial A_0(x)}{\partial x^i} - \frac{\partial v_2(x)}{\partial x^i} \tag{61}$$

$$\dot{\Pi}^0 = \frac{\partial \Pi^i(x)}{\partial x^i} - J^0(x) \tag{62}$$

$$\dot{\Pi}^i = \partial_k \partial_k A_i(x) - \partial_i \partial_k A_k(x) - J^i(x) \tag{63}$$

Now, from the classifications of the constraints in (58), we know that this vector theory has

$$\mathcal{N} = 8\ \mathit{Field\ Degrees\ of\ Freedom} \tag{64}$$

$$n_1 = 2\ \mathit{First-Class\ Constraints} \tag{65}$$

$$n_2 = 0\ \mathit{Second-Class\ Constraints} \tag{66}$$

Which means that it still has

$$\mathcal{N} - 2n_1 - n_2 = 8 - 2(2) - 0 = \ \mathbf{4}\ \mathit{unaccounted\, degrees\, of\ freedom}. \tag{67}$$

In the case of Electricity and Magnetism, these four degrees of freedom are consistent with two transverse modes of a photon, each with a corresponding conjugate momentum. With this basis of comparison now established, we turn our attention to the particular Bumblebee models under consideration.

5.2 Vector Theory with a Lagrange-Multiplier Field

5.2.1 Overview

Under consideration in this section is the bumblebee model with the potential of the form

$$V = \lambda(A_\mu A^\mu \pm b^2). \tag{68}$$

Here we not only have the dynamical vector field A_μ associated with the electromagnetic vector potential, but also an additional field λ, called a Lagrange-multiplier field. The notation used is that $a \in (0, 1, 2, 3, \lambda) = (0, i, \lambda)$ so that $A_a = (A_\mu, \lambda)$ with $A_\lambda = \lambda$. Similarly, $\Pi^a = (\Pi^\mu, \Pi^\lambda)$, where Π^λ is the generalized momentum conjugate to field λ.

Additionally, the Poisson Bracket is modified slightly: for two functions $f = f(A_a, \Pi^a)$ and $g = g(A_a, \Pi^a)$, we now have that

$$[f(x), g(y)] = \int dz \left(\frac{\partial f(x)}{\partial A_a(z)} \frac{\partial g(y)}{\partial \Pi^a(z)} - \frac{\partial f(x)}{\partial \Pi^a(z)} \frac{\partial g(y)}{\partial A_a(z)} \right). \tag{69}$$

5.2.2 Analysis

We begin with the Lagrangian Density

$$\mathcal{L} = a_1(\partial_\mu A_\nu)(\partial^\mu A^\nu) + a_2(\partial_\mu A^\mu)(\partial_\nu A^\nu) + a_3(\partial_\mu A_\nu)(\partial^\nu A^\mu) - \lambda(A_\mu A^\mu \pm b^2) \tag{70}$$

Which can be rewritten as

$$\begin{aligned}\mathcal{L} = &-a_1(\partial_i A_0)^2 - a_1(\partial_0 A_i)^2 + a_1(\partial_i A_j)^2 + a_2(\partial_i A_i)(\partial_j A_j) - 2a_2(\partial_0 A_0)(\partial_i A_i) \\ &+ a_3(\partial_i A_j)(\partial_j A_i) - 2a_3(\partial_i A_0)(\partial_0 A_i) + (a_1 + a_2 + a_3)(\partial_0 A_0)^2 - \lambda((A_0)^2 - (A_i)^2 \pm b^2).\end{aligned} \tag{71}$$

From the Lagrangian Density, we can calculate the conjugate momenta:

$$\Pi^0 = 2(a_1 + a_2 + a_3)(\partial_0 A_0) - 2a_2(\partial_i A_i) \tag{72}$$

$$\Pi^i = -2a_1(\partial_0 A_i) - 2a_3(\partial_i A_0) \tag{73}$$

$$\Pi^\lambda = 0, \tag{74}$$

and immediately recognize a Primary Constraint

$$\phi_1 = \Pi^\lambda. \tag{75}$$

As detailed above, we use the conjugate momenta to construct the Hamiltonian Density

$$\begin{aligned}\mathcal{H} = &\left(\frac{(a_1)^2 - (a_3)^2}{a_1}\right)(\partial_i A_0)^2 - \left(\frac{1}{4a_1}\right)(\Pi^i)^2 - \left(\frac{a_3}{a_1}\right)\Pi^i \partial_i A_0 - a_1(\partial_i A_j)^2 \\ &- a_2(\partial_i A_i)(\partial_j A_j) - a_3(\partial_i A_j)(\partial_j A_i) + \left(\frac{1}{4(a_1 + a_2 + a_3)}\right)(\Pi^0)^2 \\ &+ \left(\frac{a_2}{(a_1 + a_2 + a_3)}\right)\Pi^0 \partial_i A_i + \left(\frac{(a_2)^2}{(a_1 + a_2 + a_3)}\right)(\partial_i A_i)^2 + \lambda((A_0)^2 - (A_i)^2 \pm b^2),\end{aligned} \tag{76}$$

and use this Hamiltonian to investigate the time evolution of ϕ_1:

$$\dot{\phi_1} = [\phi_1(x), H] + \frac{\partial \phi_1}{\partial t} \tag{77}$$

Clearly $\frac{\partial \phi_1}{\partial t} = 0$, so the above equation reduces to

$$\dot{\phi_1} = [\phi_1(x), H] = \int dz \left(\frac{\partial \phi_1(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial \phi_1(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial A_a(z)} \right). \tag{78}$$

But $\frac{\partial \phi_1(x)}{\partial A_a(z)} = 0$, so again the above equation is simply

$$\dot{\phi_1} = [\phi_1(x), H] = -\int dz \frac{\partial \phi_1(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial A_a(z)}. \tag{79}$$

Now,

$$\frac{\partial \phi_1(x)}{\partial \Pi^a(z)} = \delta_a^\lambda \delta(z - x) \tag{80}$$

and thus we only need to worry about

$$\frac{\partial H}{\partial A_\lambda(z)} = (\big((A_0(z))^2 - (A_i(z))^2 \pm (b)^2\big)). \tag{81}$$

Thus, from the above equations, we see that

$$\dot{\phi_1} = -(\big((A_0(x))^2 - (A_i(x))^2 \pm (b)^2\big)), \tag{82}$$

And we require this to be equal to zero, to keep the time-evolution of constraints constant. Thus we have defined a new, secondary constraint:

$$\phi_2 = -((A_0)^2 - (A_i)^2 \pm b^2) \tag{83}$$

Investigating the time-evolution of this new secondary constraint, we use that

$$\dot{\phi_2} = [\phi_2(x), H_T] + \frac{\partial \phi_2}{\partial t} = [\phi_2(x), H] + \left[\phi_2(x), \int u_1(y)\phi_1(y)dy\right] + \frac{\partial \phi_2}{\partial t}. \tag{84}$$

Immediately we see that $\frac{\partial \phi_2}{\partial t} = 0$, so the above equation reduces to

$$\dot{\phi_2} = [\phi_2(x), H_T] = [\phi_2(x), H] + \left[\phi_2(x), \int u_1(y)\phi_1(y)dy\right]. \tag{85}$$

Investigating

$$[\phi_2(x), H] = \int dz \left(\frac{\partial \phi_2(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial \phi_2(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial A_a(z)} \right) \tag{86}$$

we recognize that $\frac{\partial \phi_2(x)}{\partial \Pi^a(z)} = 0$, which reduces the above equation to

$$[\phi_2(x), H] = \int dz \frac{\partial \phi_2(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)}. \tag{87}$$

Now,

$$\frac{\partial \phi_2(x)}{\partial A_a(z)} = \left(-2A_0\delta^0_a + 2A_i\delta^i_a\right)\delta(z-x) \tag{88}$$

and

$$\begin{aligned}\frac{\partial H}{\partial \Pi^a(z)} = &\left[-\left(\frac{1}{2a_1}\right)\Pi^i - \left(\frac{a_3}{a_1}\right)(\partial_i A_0)\right]\delta^i_a \\ &+\left[\left(\frac{1}{2(a_1+a_2+a_3)}\right)\Pi^0 + \left(\frac{a_2}{(a_1+a_2+a_3)}\right)(\partial_i A_i)\right]\delta^0_a.\end{aligned} \tag{89}$$

Thus, we find that

$$\begin{aligned}[\phi_2(x), H] = &-2A_i\left[\left(\frac{1}{2a_1}\right)\Pi^i + \left(\frac{a_3}{a_1}\right)(\partial_i A_0)\right] \\ &-2A_0\left[\left(\frac{1}{2(a_1+a_2+a_3)}\right)\Pi^0 + \left(\frac{a_2}{(a_1+a_2+a_3)}\right)(\partial_i A_i)\right].\end{aligned} \tag{90}$$

Now, investigating

$$\begin{aligned}\left[\phi_2(x), \int u_1(y)\phi_1(y)dy\right] &= \int dy\ u_1(y)\left[\phi_2(x), \phi_1(y)\right] \\ &= \int dz\int dy\ u_1(y)\left(\frac{\partial\phi_2(x)}{\partial A_a(z)}\frac{\partial\phi_1(y)}{\partial\Pi^a(z)} - \frac{\partial\phi_2(x)}{\partial\Pi^a(z)}\frac{\partial\phi_1(y)}{\partial A_a(z)}\right)\end{aligned} \tag{91}$$

We first recall that $\frac{\partial\phi_1(x)}{\partial A_a(z)} = 0$, which reduces the above equation to

$$\left[\phi_2(x), \int u_1(y)\phi_1(y)dy\right] = \int dz\int dy\ u_1(y)\left(\frac{\partial\phi_2(x)}{\partial A_a(z)}\frac{\partial\phi_1(y)}{\partial\Pi^a(z)}\right). \tag{92}$$

And we see quite readily that

$$\left(\frac{\partial\phi_2(x)}{\partial A_a(z)}\frac{\partial\phi_1(y)}{\partial\Pi^a(z)}\right) = \left[\left(-2A_0\delta^0_a + 2A_i\delta^i_a\right)\delta(z-x)\right]\left[\delta^\lambda_a\delta(z-y)\right] = 0. \tag{93}$$

Thus, from the information above, we know that

$$\begin{aligned}\dot{\phi_2} = &-2A_i\left[\left(\frac{1}{2a_1}\right)\Pi^i + \left(\frac{a_3}{a_1}\right)(\partial_i A_0)\right] \\ &-2A_0\left[\left(\frac{1}{2(a_1+a_2+a_3)}\right)\Pi^0 + \left(\frac{a_2}{(a_1+a_2+a_3)}\right)(\partial_i A_i)\right].\end{aligned} \tag{94}$$

We require that this time-evolution also remain constant, and thus we set the above equation equal to zero. This reduces pretty readily to:

$$\begin{aligned}&A_i\left[\left(\frac{1}{2a_1}\right)\Pi^i + \left(\frac{a_3}{a_1}\right)(\partial_i A_0)\right] \\ &+A_0\left[\left(\frac{1}{2(a_1+a_2+a_3)}\right)\Pi^0 + \left(\frac{a_2}{(a_1+a_2+a_3)}\right)(\partial_i A_i)\right] = 0.\end{aligned} \tag{95}$$

This then defines another secondary constraint

$$\begin{aligned}\phi_3 =&A_i \left[\left(\frac{1}{2a_1} \right) \Pi^i + \left(\frac{a_3}{a_1} \right) (\partial_i A_0) \right] \\ &+ A_0 \left[\left(\frac{1}{2(a_1+a_2+a_3)} \right) \Pi^0 + \left(\frac{a_2}{(a_1+a_2+a_3)} \right) (\partial_i A_i) \right].\end{aligned} \tag{96}$$

Investigating the time-evolution of this new constraint, we consider

$$\dot{\phi_3} = [\phi_3(x), H_T] + \frac{\partial \phi_3}{\partial t} = [\phi_3(x), H] + \left[\phi_3(x), \int u_1(y)\phi_1(y)dy \right] + \frac{\partial \phi_3}{\partial t}, \tag{97}$$

but immediately we recognize that $\frac{\partial \phi_3}{\partial t} = 0$, and thus the above equation is reduced to simply

$$\dot{\phi_3} = [\phi_3(x), H] + \left[\phi_3(x), \int dy u_1(y)\phi_1(y) \right]. \tag{98}$$

Let's first look at

$$\begin{aligned}\left[\phi_3(x), \int u_1(y)\phi_1(y)dy \right] &= \int dy \; u_1(y) \left[\phi_3(x), \phi_1(y) \right] \\ &= \int dz \int dy \; u_1(y) \left(\frac{\partial \phi_3(x)}{\partial A_a(z)} \frac{\partial \phi_1(y)}{\partial \Pi^a(z)} \right)\end{aligned} \tag{99}$$

since $\frac{\partial \phi_1(y)}{\partial A_a(z)} = 0$. Now, by definition

$$\frac{\partial \phi_1(y)}{\partial \Pi^a(z)} = \delta^\lambda_a \delta(z-y), \tag{100}$$

but upon inspection we see that ϕ_3 has no λ-dependence, and thus we know that

$$\left[\phi_3(x), \int u_1(y)\phi_1(y)dy \right] = \mathbf{0}. \tag{101}$$

Thus we can focus solely on

$$[\phi_3(x), H] = \int dz \left(\frac{\partial \phi_3(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial \phi_3(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial A_a(z)} \right), \tag{102}$$

taking it piecewise for comfort. First we see that

$$\begin{aligned}\frac{\partial \phi_3(x)}{\partial A_a(z)} = &\left[\left(\left(\frac{1}{2(a_1+a_2+a_3)} \right) \Pi^0 + \left(\frac{a_2}{(a_1+a_2+a_3)} \right) (\partial_i A_i) - \left(\frac{a_3}{a_1} \right) (\partial_i A_i) \right) \delta^0_a \right. \\ &\left. \left(\left(\frac{1}{2a_1} \right) \Pi^i + \left(\frac{a_3}{a_1} \right) (\partial_i A_0) - \left(\frac{a_2}{(a_1+a_2+a_3)} \right) (\partial_i A_0) \right) \delta^i_a \right] \delta(z-x).\end{aligned} \tag{103}$$

Next we find that

$$\begin{aligned}\frac{\partial H}{\partial \Pi^a(z)} = &\left[\left(\frac{1}{2(a_1+a_2+a_3)} \right) \Pi^0(z) + \left(\frac{a_2}{(a_1+a_2+a_3)} \right) (\partial_i A_i(z)) \right] \delta^0_a \\ &+ \left[- \left(\frac{1}{2a_1} \right) \Pi^i(z) - \left(\frac{a_3}{a_1} \right) (\partial_i A_0(z)) \right] \delta^i_a.\end{aligned} \tag{104}$$

Using the information above, it can be shown that

$$
\begin{aligned}
\int dz \left(\frac{\partial \phi_3(x)}{\partial A_a(z)}\frac{\partial H}{\partial \Pi^a(z)}\right) &= -\left(\frac{1}{4(a_1)^2}\right)(\Pi^i)^2 + \left(\frac{1}{4(a_1+a_2+a_3)^2}\right)(\Pi^0)^2 \\
&+ \left(\left(\frac{a_3 a_2}{a_1(a_1+a_2+a_3)}\right) - \left(\frac{(a_3)^2}{(a_1)^2}\right)\right)(\partial_i A_0)^2 \\
&+ \left(\left(\frac{(a_2)^2}{(a_1+a_2+a_3)^2}\right) - \left(\frac{a_3 a_2}{a_1(a_1+a_2+a_3)}\right)\right)(\partial_i A_i)^2 \\
&+ \left(\left(\frac{a_2}{2a_1(a_1+a_2+a_3)}\right) - \left(\frac{a_3}{(a_1)^2}\right)\right)\Pi^i \partial_i A_0 \\
&+ \left(\left(\frac{a_2}{(a_1+a_2+a_3)^2}\right) - \left(\frac{a_3}{2a_1(a_1+a_2+a_3)}\right)\right)\Pi^0 \partial_i A_i .
\end{aligned}
\tag{105}
$$

We proceed, calculating

$$
\frac{\partial \phi_3(x)}{\partial \Pi^a(z)} = \left(\frac{1}{2(a_1+a_2+a_3)}\right) A_0 \delta^0_a + \left(\frac{1}{2a_1}\right) A_i \delta^i_a
\tag{106}
$$

and

$$
\begin{aligned}
\frac{\partial H}{\partial A_a(z)} &= \left[(-2\left(\frac{(a_1)^2-(a_3)^2}{a_1}\right)\partial_i\partial_i A_0(z) + \left(\frac{a_3}{a_1}\right)\partial_i \Pi^i(z) + 2\lambda A_0(z)\right]\delta^0_a \\
&+ \left[2a_1(\partial_k\partial_k A_i(z)) + 2\left(\left(\frac{-(a_2)^2}{(a_1+a_2+a_3)}\right) + a_2 + a_3\right)(\partial_i\partial_k A_k(z)) \right. \\
&\left. - \left(\frac{a_2}{(a_1+a_2+a_3)}\right)\partial_i \Pi^0(z) - 2\lambda A_i(z)\right]\delta^i_a .
\end{aligned}
\tag{107}
$$

Using these equations above, it can be shown that

$$
\begin{aligned}
-\int dz \left(\frac{\partial \phi_3(x)}{\partial \Pi^a(z)}\frac{\partial H}{\partial A_a(z)}\right) &= -\left(\frac{\lambda}{(a_1+a_2+a_3)}\right)(A_0)^2 + \left(\frac{\lambda}{a_1}\right)(A_i)^2 \\
&- A_i\partial_j\partial_j A_i + \left(\left(\frac{(a_2)^2}{(a_1)(a_1+a_2+a_3)}\right) - \left(\frac{a_2+a_3}{a_1}\right)\right)(A_i\partial_i\partial_j A_j) \\
&+ \left(\frac{a_2}{2a_1(a_1+a_2+a_3)}\right) A_i\partial_i\Pi^0 + \left(\frac{(a_1)^2-(a_3)^2}{a_1(a_1+a_2+a_3)}\right) A_0\partial_i\partial_i A_0 \\
&- \left(\frac{a_3}{2a_1(a_1+a_2+a_3)}\right) A_0\partial_i\Pi^i .
\end{aligned}
\tag{108}
$$

Now, from all of the information above, we can see that

$$
\begin{aligned}
\dot{\phi_3} = &-\left(\frac{\lambda}{(a_1+a_2+a_3)}\right)(A_0)^2 + \left(\left(\frac{a_3a_2}{a_1(a_1+a_2+a_3)}\right) - \left(\frac{(a_3)^2}{(a_1)^2}\right)\right)(\partial_i A_0)^2 \\
&+ \left(\left(\frac{(a_2)^2}{(a_1+a_2+a_3)^2}\right) - \left(\frac{a_3a_2}{a_1(a_1+a_2+a_3)}\right)\right)(\partial_i A_i)^2 - A_i\partial_j\partial_j A_i + \left(\frac{\lambda}{a_1}\right)(A_i)^2 \\
&+ \left(\left(\frac{(a_2)^2}{(a_1)(a_1+a_2+a_3)}\right) - \left(\frac{a_2+a_3}{a_1}\right)\right)(A_i\partial_i\partial_j A_j) \\
&+ \left(\frac{(a_1)^2-(a_3)^2}{a_1(a_1+a_2+a_3)}\right)A_0\partial_i\partial_i A_0 - \left(\frac{1}{4(a_1)^2}\right)(\Pi^i)^2 \\
&+ \left(\frac{a_2}{2a_1(a_1+a_2+a_3)}\right)A_i\partial_i\Pi^0 - \left(\frac{a_3}{2a_1(a_1+a_2+a_3)}\right)A_0\partial_i\Pi^i \\
&+ \left(\left(\frac{a_2}{2a_1(a_1+a_2+a_3)}\right) - \left(\frac{a_3}{(a_1)^2}\right)\right)\Pi^i\partial_i A_0 + \left(\frac{1}{4(a_1+a_2+a_3)^2}\right)(\Pi^0)^2 \\
&+ \left(\left(\frac{a_2}{(a_1+a_2+a_3)^2}\right) - \left(\frac{a_3}{2a_1(a_1+a_2+a_3)}\right)\right)\Pi^0\partial_i A_i.
\end{aligned}
\tag{109}
$$

As before, we must ensure that the time-evolution of this constraint remains constant, and thus we set the above equation to zero. This simplifies further to

$$
\begin{aligned}
&-\lambda(A_0)^2 + \left(\frac{\lambda(a_1+a_2+a_3)}{a_1}\right)(A_i)^2 + \left(\left(\frac{a_3a_2}{a_1}\right) - \left(\frac{(a_1+a_2+a_3)(a_3)^2}{(a_1)^2}\right)\right)(\partial_i A_0)^2 \\
&+ \left(\left(\frac{(a_2)^2}{(a_1+a_2+a_3)}\right) - \left(\frac{a_3a_2}{a_1}\right)\right)(\partial_i A_i)^2 - (a_1+a_2+a_3)A_i\partial_j\partial_j A_i \\
&+ \left(\left(\frac{(a_2)^2}{(a_1)}\right) - \left(\frac{(a_1+a_2+a_3)(a_2+a_3)}{a_1}\right)\right)(A_i\partial_i\partial_j A_j) + \left(\frac{(a_1)^2-(a_3)^2}{a_1}\right)A_0\partial_i\partial_i A_0 \\
&+ \left(\frac{a_2}{2a_1}\right)A_i\partial_i\Pi^0 - \left(\frac{a_3}{2a_1}\right)A_0\partial_i\Pi^i + \left(\left(\frac{a_2}{2a_1}\right) - \left(\frac{a_3(a_1+a_2+a_3)}{(a_1)^2}\right)\right)\Pi^i\partial_i A_0 \\
&+ \left(\left(\frac{a_2}{(a_1+a_2+a_3)}\right) - \left(\frac{a_3}{2a_1}\right)\right)\Pi^0\partial_i A_i + \left(\frac{1}{4(a_1+a_2+a_3)}\right)(\Pi^0)^2 \\
&- \left(\frac{(a_1+a_2+a_3)}{4(a_1)^2}\right)(\Pi^i)^2 = \mathbf{0},
\end{aligned}
\tag{110}
$$

which defines *yet another* secondary constraint,

$$
\begin{aligned}
\phi_4 = &-\lambda(A_0)^2 + \left(\frac{\lambda(a_1+a_2+a_3)}{a_1}\right)(A_i)^2 + \left(\left(\frac{a_3 a_2}{a_1}\right) - \left(\frac{(a_1+a_2+a_3)(a_3)^2}{(a_1)^2}\right)\right)(\partial_i A_0)^2 \\
&+ \left(\left(\frac{(a_2)^2}{(a_1+a_2+a_3)}\right) - \left(\frac{a_3 a_2}{a_1}\right)\right)(\partial_i A_i)^2 - (a_1+a_2+a_3)A_i\partial_j\partial_j A_i \\
&+ \left(\left(\frac{(a_2)^2}{(a_1)}\right) - \left(\frac{(a_1+a_2+a_3)(a_2+a_3)}{a_1}\right)\right)(A_i\partial_i\partial_j A_j) + \left(\frac{(a_1)^2-(a_3)^2}{a_1}\right)A_0\partial_i\partial_i A_0 \\
&+ \left(\frac{a_2}{2a_1}\right)A_i\partial_i\Pi^0 - \left(\frac{a_3}{2a_1}\right)A_0\partial_i\Pi^i + \left(\left(\frac{a_2}{2a_1}\right) - \left(\frac{a_3(a_1+a_2+a_3)}{(a_1)^2}\right)\right)\Pi^i\partial_i A_0 \\
&+ \left(\left(\frac{a_2}{(a_1+a_2+a_3)}\right) - \left(\frac{a_3}{2a_1}\right)\right)\Pi^0\partial_i A_i + \left(\frac{1}{4(a_1+a_2+a_3)}\right)(\Pi^0)^2 \\
&- \left(\frac{(a_1+a_2+a_3)}{4(a_1)^2}\right)(\Pi^i)^2.
\end{aligned}
\tag{111}
$$

Now, as painful as it may look, we can also investigate the time-evolution of this constraint as well, according to the familiar equation

$$\dot{\phi}_4 = [\phi_4(x), H_T] + \frac{\partial\phi_4}{\partial t} = [\phi_4(x), H] + \left[\phi_4(x), \int u_1(y)\phi_1(y)dy\right] + \frac{\partial\phi_4}{\partial t}, \tag{112}$$

but again we immediately recognize that $\frac{\partial\phi_4}{\partial t} = 0$. Thus, let's first look at

$$
\begin{aligned}
\left[\phi_4(x), \int u_1(y)\phi_1(y)dy\right] &= \int dy\ u_1(y)\,[\phi_4(x), \phi_1(y)] \\
&= \int dz \int dy\ u_1(y)\left(\frac{\partial\phi_4(x)}{\partial A_a(z)}\frac{\partial\phi_1(y)}{\partial\Pi^a(z)}\right).
\end{aligned}
\tag{113}
$$

By definition

$$\frac{\partial\phi_1(y)}{\partial\Pi^a(z)} = \delta_a^\lambda\delta(z-y), \tag{114}$$

and thus we're concerned only with

$$\frac{\partial\phi_4(x)}{\partial A_\lambda(z)} = \left(\left(\frac{(a_1+a_2+a_3)}{a_1}\right)(A_i)^2 - (A_0)^2\right)\delta(z-x). \tag{115}$$

Thus, from the information above we can see that

$$\left[\phi_4(x), \int u_1(y)\phi_1(y)dy\right] = u_1(x)\left(\left(\frac{(a_1+a_2+a_3)}{a_1}\right)(A_i(x))^2 - (A_0(x))^2\right). \tag{116}$$

Now we turn our attention to

$$[\phi_4(x), H] = \int dz\left(\frac{\partial\phi_4(x)}{\partial A_a(z)}\frac{\partial H}{\partial\Pi^a(z)} - \frac{\partial\phi_4(x)}{\partial\Pi^a(z)}\frac{\partial H}{\partial A_a(z)}\right), \tag{117}$$

taking it apart piece-by-piece. First we find that

$$\begin{aligned}\frac{\partial\phi_4(x)}{\partial A_a(z)} = &\left[-2\lambda A_0 + 2\left(\left(\frac{((a_1)^2-(a_3)^2-a_3a_2)}{a_1}\right)+\left(\frac{(a_1+a_2+a_3)(a_3)^2}{(a_1)^2}\right)\right)\partial_i\partial_i A_0\right.\\ &\left.+\left(\left(\frac{a_3(a_1+a_2+a_3)}{(a_1)^2}\right)-\left(\frac{(a_3+a_2)}{2a_1}\right)\right)\partial_i\Pi^i\right]\delta_a^0+\left[2\left(\frac{(a_1+a_2+a_3)}{a_1}\right)\lambda A_i\right.\\ &-2\left(\left(\frac{a_3(a_1+a_2+a_3)}{a_1}\right)+\left(\frac{(a_2)^2}{(a_1+a_2+a_3)}\right)+a_2\right)\partial_i\partial_j A_j\\ &\left.-2(a_1+a_2+a_3)\partial_j\partial_j A_i+\left(\left(\frac{(a_2+a_3)}{2a_1}\right)-\left(\frac{a_2}{(a_1+a_2+a_3)}\right)\right)\partial_i\Pi^0\right]\delta_a^i\\ &+\left[\left(\left(\frac{(a_1+a_2+a_3)}{a_1}\right)(A_i)^2-(A_0)^2\right)\right]\delta_a^\lambda,\end{aligned} \tag{118}$$

and that

$$\begin{aligned}\frac{\partial H}{\partial\Pi^a(z)} = &\left[\left(\frac{1}{2(a_1+a_2+a_3)}\right)\Pi^0+\left(\frac{a_2}{(a_1+a_2+a_3)}\right)\partial_i A_i\right]\delta_a^0\\ &+\left[-\left(\frac{1}{2a_1}\right)\Pi^i-\left(\frac{a_3}{a_2}\right)\partial_i A_0\right]\delta_a^i.\end{aligned} \tag{119}$$

At this point we calculate

$$\begin{aligned}\int dz\,\frac{\partial\phi_4(x)}{\partial A_a(z)}\frac{\partial H}{\partial\Pi^a(z)} = &-\left(\frac{1}{(a_1+a_2+a_3)}\right)(\lambda A_0)(\Pi^0)-2\left(\frac{a_2}{(a_1+a_2+a_3)}\right)(\lambda A_0)(\partial_i A_i)\\ &+\left(\left(\frac{(a_1)^2-(a_3)^2-a_2a_3}{a_1(a_1+a_2+a_3)}\right)+\left(\frac{(a_3)^2}{(a_1)^2}\right)\right)(\partial_i\partial_i A_0)(\Pi^0)+\left(\frac{(a_1+a_2+a_3)}{a_1}\right)(\partial_j\partial_j A_i)(\Pi^i)\\ &+\left(\left(\frac{a_3}{2(a_1)^2}\right)-\left(\frac{(a_3+a_2)}{4a_1(a_1+a_2+a_3)}\right)\right)(\partial_i\Pi^i)(\Pi^0)-2\left(\frac{a_3(a_1+a_2+a_3)}{(a_1)^2}\right)(\lambda A_i)(\partial_i A_0)\\ &+2\left(\left(\frac{((a_1)^2-(a_3)^2-a_2a_3)a_2}{a_1(a_1+a_2+a_3)}\right)+\left(\frac{a_2(a_3)^2}{(a_1)^2}\right)\right)(\partial_i\partial_i A_0)(\partial_j A_j)\\ &+\left(\left(\frac{a_2a_3}{(a_1)^2}\right)-\left(\frac{(a_2+a_3)a_2}{2a_1(a_1+a_2+a_3)}\right)\right)(\partial_i\Pi^i)(\partial_j A_j)-\left(\frac{(a_1+a_2+a_3)}{(a_1)^2}\right)(\lambda A_i)(\Pi^i)\\ &+\left(\left(\frac{a_3(a_1+a_2+a_3)}{(a_1)^2}\right)+\left(\frac{(a_2)^2}{a_1(a_1+a_2+a_3)}\right)+\left(\frac{a_2}{a_1}\right)\right)(\partial_i\partial_j A_j)(\Pi^i)\\ &+\left(\left(\frac{a_2}{2a_1(a_1+a_2+a_3)}\right)-\left(\frac{(a_2+a_3)}{2(a_1)^2}\right)\right)(\partial_i\Pi^0)(\Pi^i)\\ &+2\left(\left(\frac{(a_3)^2(a_1+a_2+a_3)}{(a_1)^2}\right)+\left(\frac{(a_2)^2a_3}{a_1(a_1+a_2+a_3)}\right)+\left(\frac{a_2a_3}{a_1}\right)\right)(\partial_i\partial_j A_j)(\partial_i A_0)\\ &+2\left(\frac{a_3(a_1+a_2+a_3)}{a_1}\right)(\partial_j\partial_j A_i)(\partial_i A_0)\\ &+\left(\left(\frac{a_2a_3}{a_1(a_1+a_2+a_3)}\right)-\left(\frac{(a_2+a_3)a_3}{2(a_1)^2}\right)\right)(\partial_i\Pi^0)(\partial_i A_0).\end{aligned} \tag{120}$$

Now we look at

$$\frac{\partial\phi_4(x)}{\partial\Pi^a(z)} = \left[\left(\left(\frac{a_2}{(a_1+a_2+a_3)}\right) - \left(\frac{(a_2+a_3)}{2a_1}\right)\right)\partial_i A_i + \left(\frac{1}{2(a_1+a_2+a_3)}\right)\Pi^0\right]\delta_a^0$$
$$+ \left[\left(\left(\frac{(a_2+a_3)}{2a_1}\right) - \left(\frac{a_3(a_1+a_2+a_3)}{(a_1)^2}\right)\right)\partial_i A_0 - \left(\frac{(a_1+a_2+a_3)}{2(a_1)^2}\right)\Pi^i\right]\delta_a^i, \tag{121}$$

and thus we're only concerned with

$$\begin{aligned}\frac{\partial H}{\partial A_a(z)} = {} & \left[-2\left(\frac{(a_1)^2-(a_3)^2}{a_1}\right)\partial_i\partial_i A_0 + \left(\frac{a_3}{a_1}\right)\partial_i\Pi^i + 2\lambda A_0\right]\delta_a^0 \\ & + \left[2a_1(\partial_j\partial_j A_i) + 2\left(\left(-\frac{(a_2)^2}{(a_1+a_2+a_3)}\right) + (a_2+a_3)\right)(\partial_i\partial_j A_j)\right. \\ & \left. - \left(\frac{a_2}{(a_1+a_2+a_3)}\right)\partial_i\Pi^0 - 2\lambda A_i\right]\delta_a^i.\end{aligned} \tag{122}$$

From these equations, we find that

$$\begin{aligned} & -\int dz\, \frac{\partial\phi_4(x)}{\partial\Pi^a(z)}\frac{\partial H}{\partial A_a(z)} = -\left(\frac{1}{(a_1+a_2+a_3)}\right)(\lambda A_0)(\Pi^0) \\ & + \left(\frac{(a_1)^2-(a_3)^2}{a_1(a_1+a_2+a_3)}\right)(\partial_i\partial_i A_0)(\Pi^0) - \left(\frac{a_3}{2a_1(a_1+a_2+a_3)}\right)(\partial_i\Pi^i)(\Pi^0) \\ & + 2\left(\left(\frac{(a_2+a_3)}{2a_1}\right) - \left(\frac{a_2}{(a_1+a_2+a_3)}\right)\right)(\lambda A_0)(\partial_i A_i) \\ & + 2\left(\left(\frac{a_2((a_1)^2-(a_3)^2)}{a_1(a_1+a_2+a_3)}\right) - \left(\frac{(a_2+a_3)((a_1)^2-(a_3)^2)}{2(a_1)^2}\right)\right)(\partial_i\partial_i A_0)(\partial_j A_j) \\ & + \left(\left(\frac{a_3(a_2+a_3)}{2(a_1)^2}\right) - \left(\frac{a_2 a_3}{a_1(a_1+a_2+a_3)}\right)\right)(\partial_i\Pi^i)(\partial_j A_j) - \left(\frac{(a_1+a_2+a_3)}{(a_1)^2}\right)(\lambda A_i)(\Pi^i) \\ & + 2\left(\frac{(a_2+a_3)(a_1+a_2+a_3)-(a_2)^2}{2(a_1)^2}\right)(\partial_i\partial_j A_j)(\Pi^i) + \left(\frac{(a_1+a_2+a_3)}{a_1}\right)(\partial_j\partial_j A_i)(\Pi^i) \\ & - \left(\frac{a_2}{2(a_1)^2}\right)(\partial_i\Pi^0)(\Pi^i) + 2\left(\left(\frac{(a_2+a_3)}{2a_1}\right) - \left(\frac{a_3(a_1+a_2+a_3)}{(a_1)^2}\right)\right)(\lambda A_i)(\partial_i A_0) \\ & + 2\left(\left(\frac{a_3((a_1+a_2+a_3)(a_3+a_2)-(a_2)^2)}{(a_1)^2}\right) + \left(\frac{(a_2+a_3)(a_2)^2}{2a_1(a_1+a_2+a_3)}\right)\right. \\ & \left. - \left(\frac{(a_2+a_3)^2}{2a_1}\right)\right)(\partial_i\partial_j A_j)(\partial_i A_0) + \left(\left(\frac{2a_3(a_1+a_2+a_3)}{a_1}\right) - (a_2+a_3)\right)(\partial_j\partial_j A_i)(\partial_i A_0) \\ & + \left(\left(\frac{(a_3+a_2)a_2}{2a_1(a_1+a_2+a_3)}\right) - \left(\frac{a_3 a_2}{(a_1)^2}\right)\right)(\partial_i\Pi^0)(\partial_i A_0).\end{aligned} \tag{123}$$

Now, from these ungodly equations above, we can see that

$$
\begin{aligned}
&[\phi_4(x), H] = \int dz \left(\frac{\partial \phi_4(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial \phi_4(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial A_a(z)} \right) \\
&= \int dz \ \frac{\partial \phi_4(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \int dz \ \frac{\partial \phi_4(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial A_a(z)} \\
&= -\left(\frac{2}{(a_1+a_2+a_3)}\right)(\lambda A_0)(\Pi^0) + 2\left(\left(\frac{(a_2+a_3)}{2a_1}\right) - \left(\frac{2a_2}{(a_1+a_2+a_3)}\right)\right)(\lambda A_0)(\partial_i A_i) \\
&- 2\left(\frac{(a_1+a_2+a_3)}{(a_1)^2}\right)(\lambda A_i)(\Pi^i) + 2\left(\left(\frac{(a_2+a_3)}{2a_1}\right) - \left(\frac{2a_3(a_1+a_2+a_3)}{(a_1)^2}\right)\right)(\lambda A_i)(\partial_i A_0) \\
&+ \left(\left(\frac{(3a_3+a_2)a_2}{2a_1(a_1+a_2+a_3)}\right) - \left(\frac{(3a_2+a_3)a_3}{2(a_1)^2}\right)\right)(\partial_i \Pi^0)(\partial_i A_0) \\
&+ \left(\left(\frac{a_3(3a_2+a_3)}{2(a_1)^2}\right) - \left(\frac{(3a_3+a_2)a_2}{2a_1(a_1+a_2+a_3)}\right)\right)(\partial_i \Pi^i)(\partial_j A_j) \\
&+ \left(\left(\frac{a_3}{2(a_1)^2}\right) - \left(\frac{(3a_3+a_2)}{2a_1(a_1+a_2+a_3)}\right)\right)(\partial_i \Pi^i)(\Pi^0) \\
&+ \left(\left(\frac{a_2}{2a_1(a_1+a_2+a_3)}\right) - \left(\frac{(2a_2+a_3)}{2(a_1)^2}\right)\right)(\partial_i \Pi^0)(\Pi^i) \\
&+ 2\left[\left(\frac{a_2(2((a_1)^2-(a_3)^2) - a_3 a_2)}{a_1(a_1+a_2+a_3)}\right)\right. \\
&+ \left.\left(\frac{2(a_3)^2 a_2 - (a_2+a_3)((a_1)^2-(a_3)^2)}{2(a_1)^2}\right)\right](\partial_i \partial_i A_0)(\partial_j A_j) \\
&+ 2\left[\left(\frac{(a_3)^2(2a_1+3a_2+2a_3) + a_1 a_2 a_3}{(a_1)^2}\right) + \left(\frac{(a_2+3a_3)(a_2)^2}{2a_1(a_1+a_2+a_3)}\right)\right. \\
&- \left.\left(\frac{(a_2)^2+(a_3)^2+a_2 a_3}{2a_1}\right)\right](\partial_i \partial_j A_j)(\partial_i A_0) \\
&+ 2\left(\left(\frac{2a_3(a_1+a_2+a_3)}{a_1}\right) - \left(\frac{(a_2+a_3)}{2}\right)\right)(\partial_j \partial_j A_i)(\partial_i A_0) \\
&+ \left(\left(\frac{2((a_1)^2-(a_3)^2) - a_2 a_3}{a_1(a_1+a_2+a_3)}\right) + \left(\frac{a_3}{a_2}\right)^2\right)(\partial_i \partial_i A_0)(\Pi^0) \\
&+ 2\left(\left(\frac{a_2(2a_1+3a_3) + 2a_3(a_1+a_3)}{(a_1)^2}\right) + \left(\frac{(a_2)^2}{a_1(a_1+a_2+a_3)}\right)\right)(\partial_i \partial_j A_j)(\Pi^i) \\
&+ 2\left(\frac{(a_1+a_2+a_3)}{a_1}\right)(\partial_j \partial_j A_i)(\Pi^i).
\end{aligned}
\tag{124}
$$

Thus, we know that

$$
\begin{aligned}
\dot{\phi}_4 = &-\left(\frac{2}{(a_1+a_2+a_3)}\right)(\lambda A_0)(\Pi^0) + 2\left(\left(\frac{(a_2+a_3)}{2a_1}\right) - \left(\frac{2a_2}{(a_1+a_2+a_3)}\right)\right)(\lambda A_0)(\partial_i A_i) \\
&- 2\left(\frac{(a_1+a_2+a_3)}{(a_1)^2}\right)(\lambda A_i)(\Pi^i) + 2\left(\left(\frac{(a_2+a_3)}{2a_1}\right) - \left(\frac{2a_3(a_1+a_2+a_3)}{(a_1)^2}\right)\right)(\lambda A_i)(\partial_i A_0) \\
&+ \left(\left(\frac{(3a_3+a_2)a_2}{2a_1(a_1+a_2+a_3)}\right) - \left(\frac{(3a_2+a_3)a_3}{2(a_1)^2}\right)\right)(\partial_i \Pi^0)(\partial_i A_0) \\
&+ \left(\left(\frac{a_3(3a_2+a_3)}{2(a_1)^2}\right) - \left(\frac{(3a_3+a_2)a_2}{2a_1(a_1+a_2+a_3)}\right)\right)(\partial_i \Pi^i)(\partial_j A_j) \\
&+ \left(\left(\frac{a_3}{2(a_1)^2}\right) - \left(\frac{(3a_3+a_2)}{2a_1(a_1+a_2+a_3)}\right)\right)(\partial_i \Pi^i)(\Pi^0) \\
&+ \left(\left(\frac{a_2}{2a_1(a_1+a_2+a_3)}\right) - \left(\frac{(2a_2+a_3)}{2(a_1)^2}\right)\right)(\partial_i \Pi^0)(\Pi^i) \\
&+ 2\left[\left(\frac{a_2(2((a_1)^2-(a_3)^2)-a_3a_2)}{a_1(a_1+a_2+a_3)}\right)\right. \\
&\left.+ \left(\frac{2(a_3)^2a_2-(a_2+a_3)((a_1)^2-(a_3)^2)}{2(a_1)^2}\right)\right](\partial_i\partial_i A_0)(\partial_j A_j) \\
&+ 2\left[\left(\frac{(a_3)^2(2a_1+3a_2+2a_3)+a_1a_2a_3}{(a_1)^2}\right) + \left(\frac{(a_2+3a_3)(a_2)^2}{2a_1(a_1+a_2+a_3)}\right)\right. \\
&\left.- \left(\frac{(a_2)^2+(a_3)^2+a_2a_3}{2a_1}\right)\right](\partial_i\partial_j A_j)(\partial_i A_0) \\
&+ 2\left(\left(\frac{2a_3(a_1+a_2+a_3)}{a_1}\right) - \left(\frac{(a_2+a_3)}{2}\right)\right)(\partial_j\partial_j A_i)(\partial_i A_0) \\
&+ \left(\left(\frac{2((a_1)^2-(a_3)^2)-a_2a_3}{a_1(a_1+a_2+a_3)}\right) + \left(\frac{a_3}{a_2}\right)^2\right)(\partial_i\partial_i A_0)(\Pi^0) \\
&+ \left(\left(\frac{a_2(2a_1+3a_3)+2a_3(a_1+a_3)}{(a_1)^2}\right) + \left(\frac{(a_2)^2}{a_1(a_1+a_2+a_3)}\right)\right)(\partial_i\partial_j A_j)(\Pi^i) \\
&+ 2\left(\frac{(a_1+a_2+a_3)}{a_1}\right)(\partial_j\partial_j A_i)(\Pi^i) + u_1(x)\left(\left(\frac{(a_1+a_2+a_3)}{a_1}\right)(A_i(x))^2 - (A_0(x))^2\right),
\end{aligned}
\tag{125}
$$

which allows us to solve for the coefficient u_1:

$$
\begin{aligned}
u_1 = \Bigg[&\left(\frac{2}{(a_1+a_2+a_3)}\right)(\lambda A_0)(\Pi^0) - 2\left(\left(\frac{(a_2+a_3)}{2a_1}\right) - \left(\frac{2a_2}{(a_1+a_2+a_3)}\right)\right)(\lambda A_0)(\partial_i A_i) \\
&+2\left(\frac{(a_1+a_2+a_3)}{(a_1)^2}\right)(\lambda A_i)(\Pi^i) - 2\left(\left(\frac{(a_2+a_3)}{2a_1}\right) - \left(\frac{2a_3(a_1+a_2+a_3)}{(a_1)^2}\right)\right)(\lambda A_i)(\partial_i A_0) \\
&-\left(\left(\frac{(3a_3+a_2)a_2}{2a_1(a_1+a_2+a_3)}\right) - \left(\frac{(3a_2+a_3)a_3}{2(a_1)^2}\right)\right)(\partial_i \Pi^0)(\partial_i A_0) \\
&-\left(\left(\frac{a_3(3a_2+a_3)}{2(a_1)^2}\right) - \left(\frac{(3a_3+a_2)a_2}{2a_1(a_1+a_2+a_3)}\right)\right)(\partial_i \Pi^i)(\partial_j A_j) \\
&-\left(\left(\frac{a_3}{2(a_1)^2}\right) - \left(\frac{(3a_3+a_2)}{2a_1(a_1+a_2+a_3)}\right)\right)(\partial_i \Pi^i)(\Pi^0) \\
&-\left(\left(\frac{a_2}{2a_1(a_1+a_2+a_3)}\right) - \left(\frac{(2a_2+a_3)}{2(a_1)^2}\right)\right)(\partial_i \Pi^0)(\Pi^i) \\
&-2\left(\left(\frac{a_2(2((a_1)^2-(a_3)^2)-a_3a_2)}{a_1(a_1+a_2+a_3)}\right)\right. \\
&\left.+\left(\frac{2(a_3)^2a_2-(a_2+a_3)((a_1)^2-(a_3)^2)}{2(a_1)^2}\right)\right)(\partial_i\partial_i A_0)(\partial_j A_j) \\
&-2\left(\left(\frac{(a_3)^2(2a_1+3a_2+2a_3)+a_1a_2a_3}{(a_1)^2}\right) + \left(\frac{(a_2+3a_3)(a_2)^2}{2a_1(a_1+a_2+a_3)}\right)\right. \\
&\left.-\left(\frac{(a_2)^2+(a_3)^2+a_2a_3}{2a_1}\right)\right)(\partial_i\partial_j A_j)(\partial_i A_0) \\
&-2\left(\left(\frac{2a_3(a_1+a_2+a_3)}{a_1}\right) - \left(\frac{(a_2+a_3)}{2}\right)\right)(\partial_j\partial_j A_i)(\partial_i A_0) \\
&-\left(\left(\frac{2((a_1)^2-(a_3)^2)-a_2a_3}{a_1(a_1+a_2+a_3)}\right) + \left(\frac{a_3}{a_2}\right)^2\right)(\partial_i\partial_i A_0)(\Pi^0) \\
&-\left(\left(\frac{a_2(2a_1+3a_3)+2a_3(a_1+a_3)}{(a_1)^2}\right) + \left(\frac{(a_2)^2}{a_1(a_1+a_2+a_3)}\right)\right)(\partial_i\partial_j A_j)(\Pi^i) \\
&-2\left(\frac{(a_1+a_2+a_3)}{a_1}\right)(\partial_j\partial_j A_i)(\Pi^i)\Bigg]\left(\left(\frac{(a_1+a_2+a_3)}{a_1}\right)(A_i(x))^2 - (A_0(x))^2\right)^{-1}
\end{aligned}
\tag{126}
$$

Now that we have found all of our constraints, we can begin their classification:

$$[\phi_{\mathbf{1}}, \phi_{\mathbf{2}}] = 0 \tag{127}$$

$$[\phi_{\mathbf{1}}, \phi_{\mathbf{3}}] = 0 \tag{128}$$

$$[\phi_{\mathbf{1}}, \phi_{\mathbf{4}}] = \left(\left(\frac{(a_1+a_2+a_3)}{a_1}\right)(A_i)^2 - (A_0)^2\right)\delta(x-y) \tag{129}$$

$$[\phi_{\mathbf{2}}, \phi_{\mathbf{3}}] = \left(\left(\frac{1}{(a_1+a_2+a_3)}\right)(A_0)^2 - \left(\frac{1}{a_1}\right)(A_i)^2\right)\delta(x-y) \tag{130}$$

$$
\begin{aligned}
[\phi_{\mathbf{2}},\phi_{\mathbf{4}}] = &\left[\left(\frac{1}{(a_1+a_2+a_3)}\right)A_0\Pi^0 + \left(\frac{(a_1+a_2+a_3)}{(a_1)^2}\right)A_i\Pi^i\right.\\
&+2\left(\left(\frac{a_2}{(a_1+a_2+a_3)}\right) - \left(\frac{(a_2+a_3)}{2a_1}\right)\right)A_0\partial_iA_i\\
&\left.+2\left(\left(\frac{a_3(a_1+a_2+a_3)}{(a_1)^2}\right) - \left(\frac{(a_2+a_3)}{2a_1}\right)\right)A_i\partial_iA_0\right]\delta(x-y)
\end{aligned}
\tag{131}
$$

$$
\begin{aligned}
[\phi_{\mathbf{3}},\phi_{\mathbf{4}}] = &\left[\left(\frac{1}{(a_1+a_2+a_3)}\right)\lambda(A_0)^2 - \left(\frac{(a_1+a_2+a_3)}{(a_1)^2}\right)\lambda(A_i)^2\right.\\
&+\left(\left(\frac{(3a_2+a_3)a_3}{2(a_1)^2}\right) - \left(\frac{a_2(a_2+a_3)}{2a_1(a_1+a_2+a_3)}\right) - \left(\frac{(a_3)^2(a_1+a_2+a_3)}{(a_1)^3}\right)\right)(\partial_iA_0)^2\\
&+\left(\left(\frac{a_3(a_2+a_3)}{2(a_1)^2}\right) + \left(\frac{(a_2)^2}{(a_1+a_2+a_3)^2}\right) - \left(\frac{a_2(3a_3+a_2)}{2a_1(a_1+a_2+a_3)}\right)\right)(\partial_iA_i)^2\\
&-\left(\left(\frac{(a_1)^2-(a_3)^2-a_2a_3}{a_1(a_1+a_2+a_3)}\right) + \left(\frac{(a_3)^2}{(a_1)^2}\right)\right)A_0\partial_i\partial_iA_0\\
&+\left(\left(\frac{(a_1+a_3)(a_2+a_3)}{(a_1)^2}\right) + \left(\frac{(a_2)^2}{a_1(a_1+a_2+a_3)}\right)\right)A_i\partial_i\partial_jA_j\\
&+\left(\frac{(a_1+a_2+a_3)}{a_1}\right)A_i\partial_j\partial_jA_i + \left(\left(\frac{a_3+a_2}{4a_1(a_1+a_2+a_3)}\right) - \left(\frac{a_3}{2(a_1)^2}\right)\right)A_0\partial_i\Pi^i\\
&+\left(\left(\frac{a_2}{2a_1(a_1+a_2+a_3)}\right) - \left(\frac{a_2+a_3}{4(a_1)^2}\right)\right)A_i\partial_i\Pi^0\\
&+\left(\left(\frac{3a_2+a_3}{4(a_1)^2}\right) - \left(\frac{a_3(a_1+a_2+a_3)}{(a_1)^3}\right)\right)\Pi^i\partial_iA_0\\
&+\left(\left(\frac{a_2}{(a_1+a_2+a_3)^2}\right) - \left(\frac{3a_3+a_2}{4a_1(a_1+a_2+a_3)}\right)\right)\Pi^0\partial_iA_i\\
&\left.+\left(\frac{1}{4(a_1+a_2+a_3)^2}\right)(\Pi^0)^2 - \left(\frac{1}{4(a_1)^3}\right)(\Pi^i)^2\right]\delta(x-y).
\end{aligned}
\tag{132}
$$

Thus we see that

$$
(\phi_1,\phi_2,\phi_3,\phi_4) = \textbf{Second}-\textbf{Class Constraints}. \tag{133}
$$

Since ϕ_1 is the only primary constraint, this means that we can now construct the Extended Hamiltonian

$$
H_E = \int \mathcal{H}(y)dy + \int u_1(y)\phi_1(y)dy, \tag{134}
$$

and use it to find the field and momenta equations of motion.

We'll start with the field equations of motion. First we consider

$$
\dot{A}_0 = [A_0, H_E] = [A_0(x), H] + \left[A_0(x), \int dy\ u_1(y)\phi_1(y)\right] \tag{135}
$$

and investigate

$$[A_0(x), H] = \int dz \left(\frac{\partial A_0(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial A_0(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial A_a(z)} \right). \tag{136}$$

Immediately we see that $\frac{\partial A_0(x)}{\partial \Pi^a(z)} = 0$, which reduces the above equation to

$$[A_0(x), H] = \int dz \frac{\partial A_0(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)}. \tag{137}$$

By definition

$$\frac{\partial A_0(x)}{\partial A_a(z)} = \delta_a^0 \delta(z - x), \tag{138}$$

and so we're only concerned with

$$\frac{\partial H}{\partial \Pi^0(z)} = \left[\left(\frac{1}{2(a_1 + a_2 + a_3)} \right) \Pi^0(z) + \left(\frac{a_2}{(a_1 + a_2 + a_3)} \right) (\partial_i A_i(z)) \right]. \tag{139}$$

Thus,

$$[A_0(x), H] = \left(\frac{1}{2(a_1 + a_2 + a_3)} \right) \Pi^0(x) + \left(\frac{a_2}{(a_1 + a_2 + a_3)} \right) (\partial_i A_i(x)). \tag{140}$$

Now we look at

$$\begin{aligned} \left[A_0(x), \int dy\ u_1(y)\phi_1(y) \right] &= \int dy\ u_1(y) \left[A_0(x), \phi_1(y) \right] \\ &= \int dz \int dy\ u_1(y) \left(\frac{\partial A_0(x)}{\partial A_a(z)} \frac{\partial \phi_1(y)}{\partial \Pi^a(z)} \right) \\ &= \int dz \int dy\ u_1(y) \left[\delta_a^0 \delta(z - x) \right] \left[\delta_a^\lambda \delta(z - y) \right] = \mathbf{0}. \end{aligned} \tag{141}$$

From the equations above, we have thus shown that

$$\mathbf{\dot{A}_0} = \left(\frac{\mathbf{1}}{\mathbf{2(a_1 + a_2 + a_3)}} \right) \mathbf{\Pi^0(x)} + \left(\frac{\mathbf{a_2}}{\mathbf{(a_1 + a_2 + a_3)}} \right) (\partial_{\mathbf{i}} \mathbf{A_i(x)}). \tag{142}$$

Let's now look at

$$\dot{A}_i = [A_i, H_E] = [A_i(x), H] + \left[A_i(x), \int dy\ u_1(y)\phi_1(y) \right], \tag{143}$$

and start with

$$[A_i(x), H] = \int dz \left(\frac{\partial A_i(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial A_i(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial A_a(z)} \right). \tag{144}$$

which, since $\frac{\partial A_i(x)}{\partial \Pi^a(z)} = 0$, can immediately be reduced to

$$[A_i(x), H] = \int dz \frac{\partial A_i(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)}. \tag{145}$$

Now, by definition,
$$\frac{\partial A_i(x)}{\partial A_a(z)} = \delta^i_a \delta(z-x), \tag{146}$$
which means that we only have to concern ourselves with
$$\frac{\partial H}{\partial \Pi^i(z)} = \left[-\left(\frac{1}{2a_1}\right)\Pi^i(z) - \left(\frac{a_3}{a_1}\right)(\partial_i A_0(z))\right]\delta^i_a. \tag{147}$$
Clearly, then,
$$[A_i(x), H] = \left[-\left(\frac{1}{2a_1}\right)\Pi^i(x) - \left(\frac{a_3}{a_1}\right)(\partial_i A_0(x))\right]. \tag{148}$$
Now we look at
$$\begin{aligned}\left[A_i(x), \int dy\ u_1(y)\phi_1(y)\right] &= \int dy\ u_1(y)\left[A_i(x), \phi_1(y)\right] \\ &= \int dz \int dy\ u_1(y)\left(\frac{\partial A_i(x)}{\partial A_a(z)}\frac{\partial \phi_1(y)}{\partial \Pi^a(z)}\right) \\ &= \int dz \int dy\ u_1(y)\left[\delta^i_a\delta(z-x)\right]\left[\delta^\lambda_a\delta(z-y)\right] = \mathbf{0}.\end{aligned} \tag{149}$$
From the equations above, we have thus effectively shown that
$$\dot{\mathbf{A}}_{\mathbf{i}} = \left[-\left(\frac{\mathbf{1}}{\mathbf{2a_1}}\right)\mathbf{\Pi^i(x)} - \left(\frac{\mathbf{a_3}}{\mathbf{a_1}}\right)(\partial_{\mathbf{i}}\mathbf{A_0(x)})\right]. \tag{150}$$
We finally look at
$$\dot{A_\lambda} = [A_\lambda, H_E] = [A_\lambda(x), H] + \left[A_\lambda(x), \int dy\ u_1(y)\phi_1(y)\right], \tag{151}$$
And again start with
$$[A_\lambda(x), H] = \int dz\left(\frac{\partial A_\lambda(x)}{\partial A_a(z)}\frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial A_\lambda(x)}{\partial \Pi^a(z)}\frac{\partial H}{\partial A_a(z)}\right). \tag{152}$$
Again, we can see readily that $\frac{\partial A_\lambda(x)}{\partial \Pi^a(z)} = 0$, thus the above equation reduces to
$$[A_\lambda(x), H] = \int dz \frac{\partial A_\lambda(x)}{\partial A_a(z)}\frac{\partial H}{\partial \Pi^a(z)}. \tag{153}$$
By definition,
$$\frac{\partial A_\lambda(x)}{\partial A_a(z)} = \delta^\lambda_a \delta(z-x), \tag{154}$$
which limits our concern to
$$\frac{\partial H}{\partial \Pi^\lambda(z)} = 0; \tag{155}$$
Thus,
$$[A_\lambda(x), H] = \mathbf{0}. \tag{156}$$

Now we look at

$$\left[A_\lambda(x), \int dy\ u_1(y)\phi_1(y)\right] = \int dy\ u_1(y)\left[A_\lambda(x), \phi_1(y)\right]$$
$$= \int dz \int dy\ u_1(y)\left(\frac{\partial A_\lambda(x)}{\partial A_a(z)}\frac{\partial \phi_1(y)}{\partial \Pi^a(z)}\right)$$
$$= \int dz \int dy\ u_1(y)\left[\delta_a^\lambda \delta(z-x)\right]\left[\delta_a^\lambda \delta(z-y)\right]$$
$$= \mathbf{u_1}(\mathbf{x}). \tag{157}$$

Thus, we have effectively shown that

$$\dot{\mathbf{A}}_\lambda = \mathbf{u_1}(\mathbf{x}), \tag{158}$$

with $u_1(x)$ as defined above in (126).

Now let's start to crack at the momenta equations of motion. We'll begin with

$$\dot{\Pi}^0 = \left[\Pi^0(x), H_E\right]$$
$$= \left[\Pi^0(x), H\right] + \left[\Pi^0(x), \int dy\ u_1(y)\phi_1(y)\right]. \tag{159}$$

We'll start with

$$\left[\Pi^0(x), H\right] = \int dz\ \frac{\partial \Pi^0(x)}{\partial A_a(z)}\frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial \Pi^0(x)}{\partial \Pi^a(z)}\frac{\partial H}{\partial A_a(z)}, \tag{160}$$

but immediately we recognize that $\frac{\partial \Pi^0(x)}{\partial A_a(z)} = 0$, and thus the above equation reduces to

$$\left[\Pi^0(x), H\right] = -\int dz\ \frac{\partial \Pi^0(x)}{\partial \Pi^a(z)}\frac{\partial H}{\partial A_a(z)}. \tag{161}$$

By definition

$$\frac{\partial \Pi^0(x)}{\partial \Pi^a(z)} = \delta_a^0 \delta(z-x), \tag{162}$$

and thus we're only interested in

$$\frac{\partial H}{\partial A_0(z)} = -2\left(\frac{(a_1)^2 - (a_3)^2}{a_1}\right)\partial_i\partial_i A_0(z) + \left(\frac{a_3}{a_1}\right)\partial_i \Pi^i(z) + 2\lambda A_0(z). \tag{163}$$

Hence,

$$\left[\Pi^0(x), H\right] = 2\left(\frac{(a_1)^2 - (a_3)^2}{a_1}\right)\partial_i\partial_i A_0(x) - \left(\frac{a_3}{a_1}\right)\partial_i \Pi^i(x) - 2\lambda A_0(x). \tag{164}$$

Now we can look at

$$\left[\Pi^0(x), \int dy\ u_1(y)\phi_1(y)\right] = \int dy\ u_1(y)\left[\Pi^0(x), \phi_1(y)\right]$$
$$= -\int dz \int dy\ u_1(y)\frac{\partial \Pi^0(x)}{\partial \Pi^a(z)}\frac{\partial \phi_1(y)}{\partial A_a(z)}. \tag{165}$$

But recall that $\frac{\partial \phi_1(y)}{\partial A_a(z)} = 0$, and thus we have that

$$\left[\Pi^0(x), \int dy\ u_1(y)\phi_1(y)\right] = \mathbf{0}. \tag{166}$$

From the information above, we have thus shown that

$$\mathbf{\dot{\Pi}^0 = 2\left(\frac{(a_1)^2 - (a_3)^2}{a_1}\right)\partial_i\partial_i A_0(x) - \left(\frac{a_3}{a_1}\right)\partial_i \Pi^i(x) - 2\lambda A_0(x).} \tag{167}$$

Let's turn now to investigating

$$\begin{aligned}\dot{\Pi}^i &= \left[\Pi^i(x), H_E\right] \\ &= \left[\Pi^i(x), H\right] + \left[\Pi^i(x), \int dy\ u_1(y)\phi_1(y)\right];\end{aligned} \tag{168}$$

Again we'll begin with

$$\left[\Pi^i(x), H\right] = -\int dz\ \frac{\partial \Pi^i(x)}{\partial \Pi^a(z)}\frac{\partial H}{\partial A_a(z)}. \tag{169}$$

By definition

$$\frac{\partial \Pi^i(x)}{\partial \Pi^a(z)} = \delta^i_a \delta(z - x), \tag{170}$$

and thus we're only interested in

$$\begin{aligned}\frac{\partial H}{\partial A_i(z)} &= 2a_1(\partial_k\partial_k A_i(z)) + 2\left(\left(-\frac{(a_2)^2}{(a_1+a_2+a_3)}\right) + (a_2+a_3)\right)(\partial_i\partial_k A_k(x)) \\ &\quad - \left(\frac{a_2}{(a_1+a_2+a_3)}\right)\partial_i\Pi^0(z) - 2\lambda A_i(z).\end{aligned} \tag{171}$$

Thus,

$$\begin{aligned}\left[\Pi^i(x), H\right] &= -2a_1(\partial_k\partial_k A_i(x)) + 2\left(\left(\frac{(a_2)^2}{(a_1+a_2+a_3)}\right) - (a_2+a_3)\right)(\partial_i\partial_k A_k(x)) \\ &\quad + \left(\frac{a_2}{(a_1+a_2+a_3)}\right)\partial_i\Pi^0(x) + 2\lambda A_i(x).\end{aligned} \tag{172}$$

Next we consider

$$\begin{aligned}\left[\Pi^i(x), \int dy\ u_1(y)\phi_1(y)\right] &= \int dy\ u_1(y)\left[\Pi^i(x), \phi_1(y)\right] \\ &= -\int dz \int dy\ u_1(y)\frac{\partial \Pi^i(x)}{\partial \Pi^a(z)}\frac{\partial \phi_1(y)}{\partial A_a(z)}.\end{aligned} \tag{173}$$

But again, we know that $\frac{\partial \phi_1(y)}{\partial A_a(z)} = 0$, and thus we have that

$$\left[\Pi^i(x), \int dy\ u_1(y)\phi_1(y)\right] = \mathbf{0}. \tag{174}$$

From the information above, we have effectively shown that

$$\begin{aligned}\boldsymbol{\dot{\Pi}^i} = & -\boldsymbol{2a_1(\partial_k\partial_k A_i(x)) + 2\left(\left(\frac{(a_2)^2}{(a_1+a_2+a_3)}\right) - (a_2+a_3)\right)(\partial_i\partial_k A_k(x))} \\ & \boldsymbol{+\left(\frac{a_2}{(a_1+a_2+a_3)}\right)\partial_i\Pi^0(x) + 2\lambda A_i(x).}\end{aligned} \tag{175}$$

Lastly, we turn our attention to

$$\begin{aligned}\dot{\Pi}^\lambda &= \left[\Pi^\lambda(x), H_E\right] \\ &= \left[\Pi^\lambda(x), H\right] + \left[\Pi^\lambda(x), \int dy\ u_1(y)\phi_1(y)\right]\end{aligned} \tag{176}$$

Starting with

$$\left[\Pi^\lambda(x), H\right] = -\int dz\ \frac{\partial\Pi^\lambda(x)}{\partial\Pi^a(z)}\frac{\partial H}{\partial A_a(z)}, \tag{177}$$

we know that

$$\frac{\partial\Pi^\lambda(x)}{\partial\Pi^a(z)} = \delta^\lambda_a\delta(z-x) \tag{178}$$

by definition, and thus we're only concerned with

$$\frac{\partial H}{\partial A_\lambda(z)} = \left((A_0(z))^2 - (A_i(z))^2 \pm (b)^2\right). \tag{179}$$

It is thus easy to show that

$$\left[\Pi^\lambda(x), H\right] = -\left((A_0(x))^2 - (A_i(x))^2 \pm (b)^2\right). \tag{180}$$

Now we can look at

$$\begin{aligned}\left[\Pi^\lambda(x), \int dy\ u_1(y)\phi_1(y)\right] &= \int dy\ u_1(y)\left[\Pi^\lambda(x), \phi_1(y)\right] \\ &= -\int dz\int dy\ u_1(y)\frac{\partial\Pi^\lambda(x)}{\partial\Pi^a(z)}\frac{\partial\phi_1(y)}{\partial A_a(z)}.\end{aligned} \tag{181}$$

But, just as in the two previous momenta derivations, we know that $\frac{\partial\phi_1(y)}{\partial A_a(z)} = 0$, and thus we have again that

$$\left[\Pi^i(x), \int dy\ u_1(y)\phi_1(y)\right] = \mathbf{0}. \tag{182}$$

And thus, pulling all of this together, we have the final momentum equation of motion:

$$\boldsymbol{\dot{\Pi}}^\lambda = -\left((\mathbf{A_0}(\mathbf{x}))^{\mathbf{2}} - (\mathbf{A_i}(\mathbf{x}))^{\mathbf{2}} \pm (\mathbf{b})^{\mathbf{2}}\right). \tag{183}$$

Now, from the classifications of the constraints in (133), we know that this vector theory has

$$\mathcal{N} = 10\ \mathit{Field\ Degrees\ of\ Freedom} \tag{184}$$

$$n_1 = 0\ \mathit{First-Class\ Constraints} \tag{185}$$

$$n_2 = 4\ \mathit{Second-Class\ Constraints} \tag{186}$$

Which means that it still has

$$\mathcal{N} - 2n_1 - n_2 = 10 - 2(0) - 4 = \ \mathbf{6}\ unaccounted\ degrees\ of\ freedom. \tag{187}$$

So how can we properly account for these physical degrees of freedom in our theory? Four of them can be considered as the resulting massless Nambu-Goldstone Modes, which behave similarly to the two transverse photon modes and their respective conjugate momenta as discussed in the section above on Electromagnetism. However, this still leaves two degrees of freedom curiously unaccounted for. If we look at (75), we see that it has effectively constrained away Π^{λ}, and a similar investigation of the secondary constraint ϕ_2 allows for a straightforward solution of (83) for A_0, effectively constraining away the temporal field component. Similarly, ϕ_3 allows for a straightforward solution of (96) with respect to Π^0, and ϕ_4 can be used to constrain away the Lagrange multiplier field $A_\lambda = \lambda$ entirely via (111). In this manner, we can use the four constraints in this model to solve for all of the field and momenta components except for A_j and Π^j, and thus can posit that the extra two degrees of freedom specifically concern these components, representing an additional mode. This mode could potentially be massive, as is considered in [6], but it could also potentially prove to be a propagating ghost mode; these scenarios each depend heavily on specific choices of initial conditions and the coefficients a_1, a_2, a_3. Nonetheless, it is important to remark that this theory does not simply reduce to that of Electromagnetism, as is evidenced in the different constraint structure and degrees of freedom between the two theories. Although certainly similar to E&M, it is endowed with an extra mode that depends on selections of the arbitrary coefficients a_1, a_2, and a_3.

5.3 Vector Theory with a Lagrange-Multiplier Field (Linearized Approximation)

5.3.1 Overview

To examine the propagating modes in a theory, it is often sufficient to work with linearized equations where we only consider small excitations in the components. Since the bumblebee model is highly nonlinear because of the constraint imposed in the potential ($A_\mu A^\mu \pm b^2 = 0$), it's useful to examine it in a linearized limit and to investigate the effects of such a linearization on Hamilton's equations of motion for the system. Here we consider the field components as basic vectors with minor excitations ε:

$$A_\mu = b_\mu + \varepsilon_\mu \tag{188}$$

for the specific case of a *timelike* vector

$$b_\mu = (b, 0, 0, 0). \tag{189}$$

In this case, we see that

$$\begin{aligned} (A_\mu A^\mu - b^2) &= (b_\mu + \varepsilon_\mu)(b^\mu + \varepsilon^\mu) - b^2 \\ &= b_\mu b^\mu + 2b^\mu \varepsilon_\mu + \varepsilon_\mu \varepsilon^\mu - b^2. \end{aligned} \tag{190}$$

From (189) above, we know that

$$b_\mu b^\mu = b^2, \tag{191}$$

and

$$b^\mu \varepsilon_\mu = b\varepsilon_0. \tag{192}$$

Also, since we are effectively *linearizing* the potential and are only considering *small* (first-order) oscillations, we can effectively say that

$$\varepsilon_\mu \varepsilon^\mu \approx 0. \tag{193}$$

From the above equations, we have thus effectively shown that the linear approximation to the constraint in the potential is given by

$$(A_\mu A^\mu \pm b^2) \approx 2b\varepsilon_0. \tag{194}$$

In this section we are considering the linearized approximation of the bumblebee model with the Lagrange-Multiplier potential:

$$V = \lambda(A_\mu A^\mu \pm b^2) \approx 2b\lambda\varepsilon_0. \tag{195}$$

5.3.2 Analysis

We thus start with the linearized form of the Lagrangian for the case of a timelike vector in (189):

$$\begin{aligned} \mathcal{L} = &-a_1(\partial_i\varepsilon_0)^2 - a_1(\partial_0\varepsilon_i)^2 + a_1(\partial_i\varepsilon_j)^2 + a_2(\partial_i\varepsilon_i)(\partial_j\varepsilon_j) - 2a_2(\partial_0\varepsilon_0)(\partial_i\varepsilon_i) \\ &+ a_3(\partial_i\varepsilon_j)(\partial_j\varepsilon_i) - 2a_3(\partial_i\varepsilon_0)(\partial_0\varepsilon_i) + (a_1+a_2+a_3)(\partial_0\varepsilon_0)^2 - 2b\lambda\varepsilon_0, \end{aligned} \tag{196}$$

and again calculate the conjugate momenta:

$$\Pi^0 = 2(a_1 + a_2 + a_3)(\partial_0 \varepsilon_0) - 2a_2(\partial_i \varepsilon_i) \tag{197}$$

$$\Pi^i = -2a_1(\partial_0 \varepsilon_i) - 2a_3(\partial_i \varepsilon_0) \tag{198}$$

$$\Pi^\lambda = 0. \tag{199}$$

Again here we can immediately identify a Primary Constraint

$$\phi_{\mathbf{1}} = \mathbf{\Pi}^\lambda. \tag{200}$$

We then construct the Hamiltonian:

$$\begin{aligned}\mathcal{H} = &\left(\frac{(a_1)^2 - (a_3)^2}{a_1}\right)(\partial_i \varepsilon_0)^2 - \left(\frac{1}{4a_1}\right)(\Pi^i)^2 - \left(\frac{a_3}{a_1}\right)\Pi^i \partial_i \varepsilon_0 - a_1(\partial_i \varepsilon_j)^2 - a_2(\partial_i \varepsilon_i)(\partial_j \varepsilon_j) \\ &- a_3(\partial_i \varepsilon_j)(\partial_j \varepsilon_i) + \left(\frac{1}{4(a_1 + a_2 + a_3)}\right)(\Pi^0)^2 + \left(\frac{a_2}{(a_1 + a_2 + a_3)}\right)\Pi^0 \partial_i \varepsilon_i \\ &+ \left(\frac{(a_2)^2}{(a_1 + a_2 + a_3)}\right)(\partial_i \varepsilon_i)^2 + 2b\lambda\varepsilon_0,\end{aligned} \tag{201}$$

and put it to use investigating the time-evolution of ϕ_1:

$$\dot{\phi_1} = [\phi_1(x), H] + \frac{\partial \phi_1}{\partial t}. \tag{202}$$

Clearly, again, $\frac{\partial \phi_1}{\partial t} = 0$, so the above equation reduces to:

$$\dot{\phi_1} = [\phi_1(x), H] = \int dz \left(\frac{\partial \phi_1(x)}{\partial \varepsilon_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial \phi_1(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial \varepsilon_a(z)}\right), \tag{203}$$

but $\frac{\partial \phi_1(x)}{\partial \varepsilon_a(z)} = 0$, so again the above equation is simply

$$\dot{\phi_1} = [\phi_1(x), H] = -\int dz \frac{\partial \phi_1(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial \varepsilon_a(z)}. \tag{204}$$

Now, by definition,

$$\frac{\partial \phi_1(x)}{\partial \Pi^a(z)} = \delta_a^\lambda \delta(z - x) \tag{205}$$

and thus we only need to worry about

$$\frac{\partial H}{\partial A_\lambda(z)} = 2b\varepsilon_0(z). \tag{206}$$

Thus, from the above equations, we see that

$$\dot{\phi_1} = -2b\varepsilon_0(x) \tag{207}$$

And we require this to be equal to zero, to keep the time-evolution of constraints constant. However, the constant b is assumed to be non-zero–otherwise the solution is trivial–and thus we have that

$$\varepsilon_0 = 0. \tag{208}$$

This defines a new, secondary constraint

$$\phi_{\mathbf{2}} = \varepsilon_{\mathbf{0}}. \tag{209}$$

Investigating the time-evolution of this new secondary constraint, we use that

$$\dot{\phi_2} = [\phi_2(x), H_T] + \frac{\partial \phi_2}{\partial t} = [\phi_2(x), H] + \left[\phi_2(x), \int u_1(y)\phi_1(y)dy\right] + \frac{\partial \phi_2}{\partial t}. \tag{210}$$

Immediately we see that $\frac{\partial \phi_2}{\partial t} = 0$, so the above equation reduces to

$$\dot{\phi_2} = [\phi_2(x), H_T] = [\phi_2(x), H] + \left[\phi_2(x), \int u_1(y)\phi_1(y)dy\right]. \tag{211}$$

Investigating

$$[\phi_2(x), H] = \int dz \left(\frac{\partial \phi_2(x)}{\partial \varepsilon_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial \phi_2(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial \varepsilon_a(z)} \right) \tag{212}$$

we recognize that $\frac{\partial \phi_2(x)}{\partial \Pi^a(z)} = 0$, which again reduces the above equation to

$$[\phi_2(x), H] = \int dz \frac{\partial \phi_2(x)}{\partial \varepsilon_a(z)} \frac{\partial H}{\partial \Pi^a(z)}. \tag{213}$$

Now,

$$\frac{\partial \phi_2(x)}{\partial \varepsilon_a(z)} = \delta_a^0 \delta(z - x) \tag{214}$$

which means we can focus on

$$\frac{\partial H}{\partial \Pi^0(z)} = \left(\frac{1}{2(a_1 + a_2 + a_3)} \right) \Pi^0(z) + \left(\frac{a_2}{(a_1 + a_2 + a_3)} \right) (\partial_i \varepsilon_i(z)). \tag{215}$$

Clearly then

$$[\phi_2(x), H] = \left(\frac{1}{2(a_1 + a_2 + a_3)} \right) \Pi^0(x) + \left(\frac{a_2}{(a_1 + a_2 + a_3)} \right) (\partial_i \varepsilon_i(x)). \tag{216}$$

Now, investigating

$$\begin{aligned} \left[\phi_2(x), \int u_1(y)\phi_1(y)dy\right] &= \int dy\ u_1(y) \left[\phi_2(x), \phi_1(y)\right] \\ &= \int dz \int dy\ u_1(y) \left(\frac{\partial \phi_2(x)}{\partial \varepsilon_a(z)} \frac{\partial \phi_1(y)}{\partial \Pi^a(z)} - \frac{\partial \phi_2(x)}{\partial \Pi^a(z)} \frac{\partial \phi_1(y)}{\partial \varepsilon_a(z)} \right) \end{aligned} \tag{217}$$

We first recall that $\frac{\partial \phi_1(x)}{\partial \varepsilon_a(z)} = 0$, which reduces the above equation to

$$\left[\phi_2(x), \int u_1(y)\phi_1(y)dy\right] = \int dz \int dy\ u_1(y) \left(\frac{\partial \phi_2(x)}{\partial \varepsilon_a(z)} \frac{\partial \phi_1(y)}{\partial \Pi^a(z)} \right). \tag{218}$$

We see quite readily that

$$\left(\frac{\partial\phi_2(x)}{\partial\varepsilon_a(z)}\frac{\partial\phi_1(y)}{\partial\Pi^a(z)}\right) = \left[\delta_a^0\delta(z-x)\right]\left[\delta_a^\lambda\delta(z-y)\right] = 0. \tag{219}$$

Thus, from the information above, we know that

$$\dot{\phi}_2 = \left(\frac{1}{2(a_1+a_2+a_3)}\right)\Pi^0(x) + \left(\frac{a_2}{(a_1+a_2+a_3)}\right)(\partial_i\varepsilon_i(x)) \tag{220}$$

We again set this equal to zero, requiring that the time evolution of our constraints are constant. This can then be simplified to

$$\Pi^0 + 2a_2\partial_i\varepsilon_i = 0, \tag{221}$$

which defines another secondary constraint

$$\boldsymbol{\phi_3 = \Pi^0 + 2a_2\partial_i\varepsilon_i}. \tag{222}$$

We again investigate the time-evolution of this constraint according to

$$\dot{\phi}_3 = [\phi_3(x), H_T] + \frac{\partial\phi_3}{\partial t} = [\phi_3(x), H] + \left[\phi_3(x), \int u_1(y)\phi_1(y)dy\right] + \frac{\partial\phi_3}{\partial t}. \tag{223}$$

Immediately we recognize that $\frac{\partial\phi_3}{\partial t} = 0$, and thus we turn our attention first to

$$[\phi_3(x), H] = \int dz\left(\frac{\partial\phi_3(x)}{\partial\varepsilon_a(z)}\frac{\partial H}{\partial\Pi^a(z)} - \frac{\partial\phi_3(x)}{\partial\Pi^a(z)}\frac{\partial H}{\partial\varepsilon_a(z)}\right). \tag{224}$$

By definition

$$\frac{\partial\phi_3(x)}{\partial\varepsilon_a(z)} = 2a_2\frac{\partial}{\partial x^i}\delta_a^i\delta(z-x), \tag{225}$$

and thus we see that need only worry about

$$\frac{\partial H}{\partial\Pi^i(z)} = -\left(\frac{1}{2a_1}\right)\Pi^i(z) - \left(\frac{a_3}{a_1}\right)(\partial_i\varepsilon_0(z)). \tag{226}$$

Further, we know that

$$\frac{\partial\phi_3(x)}{\partial\Pi^a(z)} = \delta_a^0\delta(z-x), \tag{227}$$

so this time we're only concerned with

$$\frac{\partial H}{\partial\varepsilon_0(z)} = -2\left(\frac{(a_1)^2-(a_3)^2}{a_1}\right)\partial_i\partial_i\varepsilon_0(z) + \left(\frac{a_3}{a_1}\right)\partial_i\Pi^i(z) + 2b\lambda. \tag{228}$$

Pulling this all together we have

$$
\begin{aligned}
[\phi_3(x), H] &= \int dz \left(\frac{\partial \phi_3(x)}{\partial \varepsilon_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial \phi_3(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial \varepsilon_a(z)} \right) \\
&= \int dz \; \frac{\partial \phi_3(x)}{\partial \varepsilon_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \int dz \; \frac{\partial \phi_3(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial \varepsilon_a(z)} \\
&= \int dz \; 2a_2 \frac{\partial}{\partial x^i} \left(- \left(\frac{1}{2a_1} \right) \Pi^i(z) - \left(\frac{a_3}{a_1} \right) (\partial_i \varepsilon_0(z)) \right) \delta(z-x) \\
&- \int dz \left(-2 \left(\frac{(a_1)^2 - (a_3)^2}{a_1} \right) \partial_i \partial_i \varepsilon_0(z) + \left(\frac{a_3}{a_1} \right) \partial_i \Pi^i(z) + 2b\lambda \right) \delta(z-x) \\
&= 2a_2 \frac{\partial}{\partial x^i} \int dz \; \left(- \left(\frac{1}{2a_1} \right) \Pi^i(z) - \left(\frac{a_3}{a_1} \right) (\partial_i \varepsilon_0(z)) \right) \delta(z-x) \\
&- \left(-2 \left(\frac{(a_1)^2 - (a_3)^2}{a_1} \right) \partial_i \partial_i \varepsilon_0(x) + \left(\frac{a_3}{a_1} \right) \partial_i \Pi^i(x) + 2b\lambda \right) \\
&= 2a_2 \frac{\partial}{\partial x^i} \left(- \left(\frac{1}{2a_1} \right) \Pi^i(x) - \left(\frac{a_3}{a_1} \right) (\partial_i \varepsilon_0(x)) \right) \\
&- \left(-2 \left(\frac{(a_1)^2 - (a_3)^2}{a_1} \right) \partial_i \partial_i \varepsilon_0(x) + \left(\frac{a_3}{a_1} \right) \partial_i \Pi^i(x) + 2b\lambda \right) \\
&= - \left(\frac{a_2}{a_1} \right) \partial_i \Pi^i(x) - 2 \left(\frac{a_2 a_3}{a_1} \right) \partial_i \partial_i \varepsilon_0(x) + 2 \left(\frac{(a_1)^2 - (a_3)^2}{a_1} \right) \partial_i \partial_i \varepsilon_0(x) \\
&- \left(\frac{a_3}{a_1} \right) \partial_i \Pi^i(x) - 2b\lambda \\
&= - \left(\frac{a_2 + a_3}{a_1} \right) \partial_i \Pi^i(x) + 2 \left(\frac{(a_1)^2 - (a_3)^2 - a_3 a_2}{a_1} \right) \partial_i \partial_i \varepsilon_0(x) - 2b\lambda.
\end{aligned} \tag{229}
$$

Now we turn to investigate

$$
\begin{aligned}
\left[\phi_3(x), \int u_1(y) \phi_1(y) dy \right] &= \int dy \; u_1(y) \left[\phi_3(x), \phi_1(y) \right] \\
&= \int dz \int dy \; u_1(y) \left(\frac{\partial \phi_3(x)}{\partial \varepsilon_a(z)} \frac{\partial \phi_1(y)}{\partial \Pi^a(z)} \right) \\
&= \int dz \int dy \; u_1(y) \left[\left(2a_2 \frac{\partial}{\partial x^i} \delta^i_a \delta(z-x) \right) \left(\delta^\lambda_a \delta(z-y) \right) \right] \\
&= 0.
\end{aligned} \tag{230}
$$

From the information above, we conclude that

$$
\dot{\phi_3} = - \left(\frac{a_2 + a_3}{a_1} \right) \partial_i \Pi^i(x) + 2 \left(\frac{(a_1)^2 - (a_3)^2 - a_3 a_2}{a_1} \right) \partial_i \partial_i \varepsilon_0(x) - 2b\lambda. \tag{231}
$$

Again, we set this equal to zero to ensure that the constraint's time-evolution remains constant; this can be simplified to

$$
(a_3 + a_2) \partial_i \Pi^i - 2((a_1)^2 - (a_3)^2 - a_2 a_3) \partial_i \partial_i \varepsilon_0 + 2b\lambda a_1 = 0. \tag{232}
$$

The equation above defines yet *another* secondary constraint:

$$\phi_4 = (\mathbf{a_3} + \mathbf{a_2})\partial_\mathbf{i}\mathbf{\Pi^i} - \mathbf{2}((\mathbf{a_1})^\mathbf{2} - (\mathbf{a_3})^\mathbf{2} - \mathbf{a_2}\mathbf{a_3})\partial_\mathbf{i}\partial_\mathbf{i}\varepsilon_\mathbf{0} + \mathbf{2b}\lambda\mathbf{a_1}. \tag{233}$$

Again, we investigate the time-evolution of this secondary constraint according to

$$\dot{\phi}_4 = [\phi_4(x), H_T] + \frac{\partial\phi_4}{\partial t} = [\phi_4(x), H] + \left[\phi_4(x), \int u_1(y)\phi_1(y)dy\right] + \frac{\partial\phi_4}{\partial t}. \tag{234}$$

Immediately we recognize that $\frac{\partial\phi_4}{\partial t} = 0$, and thus we turn our attention first to

$$[\phi_4(x), H] = \int dz \left(\frac{\partial\phi_4(x)}{\partial\varepsilon_a(z)}\frac{\partial H}{\partial\Pi^a(z)} - \frac{\partial\phi_4(x)}{\partial\Pi^a(z)}\frac{\partial H}{\partial\varepsilon_a(z)}\right). \tag{235}$$

We first calculate

$$\frac{\partial\phi_4(x)}{\partial\varepsilon_a(z)} = \left(-2((a_1)^2 - (a_3)^2 - a_2a_3)\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}\delta_a^0 + 2ba_1\delta_a^\lambda\right)\delta(z-x) \tag{236}$$

followed by

$$\frac{\partial H}{\partial\Pi^a(z)} = \left(\left(\left(\frac{1}{2(a_1+a_2+a_3)}\right)\Pi^0(z) + \left(\frac{a_2}{(a_1+a_2+a_3)}\right)(\partial_i\varepsilon_i(z))\right)\delta_a^0 + 2b\varepsilon_0(z)\delta_a^\lambda\right). \tag{237}$$

At this point we find that

$$\begin{aligned}\int dz\,\frac{\partial\phi_4(x)}{\partial\varepsilon_a(z)}\frac{\partial H}{\partial\Pi^a(z)} &= \left(\frac{(a_3)^2-(a_1)^2+a_2a_3}{(a_1+a_2+a_3)}\right)\partial_i\partial_i\Pi^0(x)\\ &\quad + 2\left(\frac{(a_2)((a_3)^2-(a_1)^2+a_2a_3)}{(a_1+a_2+a_3)}\right)\partial_i\partial_i\partial_j\varepsilon_j(x) + 4b^2a_1\varepsilon_0(x).\end{aligned} \tag{238}$$

Next we calculate

$$\frac{\partial\phi_4(x)}{\partial\Pi^a(z)} = \left((a_3+a_2)\frac{\partial}{\partial x^i}\delta_a^i\right), \tag{239}$$

and thus we're only concerned here with

$$\begin{aligned}\frac{\partial H}{\partial\varepsilon_i(z)} &= 2a_1(\partial_k\partial_k\varepsilon_i(z)) + 2\left(\left(\frac{-(a_2)^2}{(a_1+a_2+a_3)}\right) + a_2 + a_3\right)(\partial_i\partial_k\varepsilon_k(z))\\ &\quad - \left(\frac{a_2}{(a_1+a_2+a_3)}\right)\partial_i\Pi^0(z).\end{aligned} \tag{240}$$

Now, putting this together, we find that

$$\begin{aligned}\int dz\,\frac{\partial\phi_4(x)}{\partial\Pi^a(z)}\frac{\partial H}{\partial\varepsilon_a(z)} &= 2(a_2+a_3)\left(\left(\frac{-(a_2)^2}{(a_1+a_2+a_3)}\right) + a_1 + a_2 + a_3\right)(\partial_i\partial_i\partial_j\varepsilon_j(z))\\ &\quad - \left(\frac{(a_2)(a_3+a_2)}{(a_1+a_2+a_3)}\right)\partial_i\partial_i\Pi^0(z).\end{aligned} \tag{241}$$

Thus we see that

$$
\begin{aligned}
[\phi_4(x), H] &= \int dz \left(\frac{\partial \phi_4(x)}{\partial \varepsilon_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial \phi_4(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial \varepsilon_a(z)} \right) \\
&= \int dz \ \frac{\partial \phi_4(x)}{\partial \varepsilon_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \int dz \ \frac{\partial \phi_4(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial \varepsilon_a(z)} \\
&= \left(\frac{(a_3)^2 - (a_1)^2 + a_2 a_3}{(a_1 + a_2 + a_3)} \right) \partial_i \partial_i \Pi^0(x) \\
&+ 2 \left(\frac{(a_2)((a_3)^2 - (a_1)^2 + a_2 a_3)}{(a_1 + a_2 + a_3)} \right) \partial_i \partial_i \partial_j \varepsilon_j(x) + 4b^2 a_1 \varepsilon_0(x) \\
&+ 2(a_2 + a_3) \left(\left(\frac{(a_2)^2}{(a_1 + a_2 + a_3)} \right) - a_1 - a_2 - a_3 \right) (\partial_i \partial_i \partial_j \varepsilon_j(x)) \\
&+ \left(\frac{(a_2)(a_3 + a_2)}{(a_1 + a_2 + a_3)} \right) \partial_i \partial_i \Pi^0(x). \\
&= 2 \left(\left(\frac{(a_2)((a_3)^2 + (a_2)^2 - (a_1)^2 + 2a_2 a_3)}{(a_1 + a_2 + a_3)} \right) - (a_1 + a_2 + a_3)(a_2 + a_3) \right) \\
&(\partial_i \partial_i \partial_j \varepsilon_j(x)) + \left(\frac{((a_3)^2 + (a_2)^2 - (a_1)^2 + 2a_2 a_3)}{(a_1 + a_2 + a_3)} \right) (\partial_i \partial_i \Pi^0(x)) + 4b^2 a_1 \varepsilon_0(x).
\end{aligned}
\tag{242}
$$

Now we are left to investigate

$$
\begin{aligned}
\left[\phi_4(x), \int u_1(y) \phi_1(y) dy \right] &= \int dy \ u_1(y) \left[\phi_4(x), \phi_1(y) \right] \\
&= \int dz \int dy \ u_1(y) \left(\frac{\partial \phi_4(x)}{\partial \varepsilon_a(z)} \frac{\partial \phi_1(y)}{\partial \Pi^a(z)} \right) \\
&= \int dz \int dy \ u_1(y) \left[\left(-2a_1((a_1)^2 - (a_3)^2 - a_2 a_3) \partial_i \partial_i \delta_a^0 \right. \right. \\
&\left. \left. + 2ba_1 \delta_a^\lambda \delta(z - x) \right) \left(\delta_a^\lambda \delta(z - y) \right) \right] \\
&= 2ba_1 u_1(x).
\end{aligned}
\tag{243}
$$

Thus we see that

$$
\begin{aligned}
\dot{\phi_4} = 2 &\left(\left(\frac{(a_2)((a_3)^2 + (a_2)^2 - (a_1)^2 + 2a_2 a_3)}{(a_1 + a_2 + a_3)} \right) - (a_1 + a_2 + a_3)(a_2 + a_3) \right) (\partial_i \partial_i \partial_j \varepsilon_j(x)) \\
+ &\left(\frac{((a_3)^2 + (a_2)^2 - (a_1)^2 + 2a_2 a_3)}{(a_1 + a_2 + a_3)} \right) (\partial_i \partial_i \Pi^0(x)) + 4b^2 a_1 \varepsilon_0(x) + 2ba_1 u_1(x);
\end{aligned}
\tag{244}
$$

this allows us to solve for the coefficient u_1:

$$
\begin{aligned}
u_1(x) = &\left(\left(\frac{(a_1 + a_2 + a_3)(a_2 + a_3)}{(ba_1)} \right) - \left(\frac{(a_2)((a_3)^2 + (a_2)^2 - (a_1)^2 + 2a_2 a_3)}{(ba_1)(a_1 + a_2 + a_3)} \right) \right) (\partial_i \partial_i \partial_j \varepsilon_j) \\
- &\left(\frac{((a_3)^2 + (a_2)^2 - (a_1)^2 + 2a_2 a_3)}{(2ba_1)(a_1 + a_2 + a_3)} \right) (\partial_i \partial_i \Pi^0) - 2b\varepsilon_0.
\end{aligned}
\tag{245}
$$

Now that we have found all of our constraints, we can begin their classification:

$$[\phi_1, \phi_2] = 0 \tag{246}$$
$$[\phi_1, \phi_3] = 0 \tag{247}$$
$$[\phi_1, \phi_4] = -2ba_1\delta(x-y) \tag{248}$$
$$[\phi_2, \phi_3] = \delta(x-y) \tag{249}$$
$$[\phi_2, \phi_4] = 0 \tag{250}$$
$$[\phi_3, \phi_4] = 2((a_3)^2 - (a_2)^2 - (a_1)^2)\partial_i\partial_i\delta(x-y) \tag{251}$$

Thus we see that

$$(\phi_1, \phi_2, \phi_3, \phi_4) = \mathbf{Second - Class\ Constraints}, \tag{252}$$

And since ϕ_1 is the only primary constraint, we can calculate the equations of motion using the Extended Hamiltonian:

$$H_E = \int dx\ \mathcal{H}(x) + \int dx\ u_1(x)\phi_1(x). \tag{253}$$

We'll start with the field equations of motion. First we consider

$$\dot{\varepsilon_0} = [\varepsilon_0, H_E] = [\varepsilon_0(x), H] + \left[\varepsilon_0(x), \int dy\ u_1(y)\phi_1(y)\right] \tag{254}$$

and investigate

$$[\varepsilon_0(x), H] = \int dz \left(\frac{\partial\varepsilon_0(x)}{\partial\varepsilon_a(z)}\frac{\partial H}{\partial\Pi^a(z)} - \frac{\partial\varepsilon_0(x)}{\partial\Pi^a(z)}\frac{\partial H}{\partial\varepsilon_a(z)}\right). \tag{255}$$

Immediately we see that $\frac{\partial\varepsilon_0(x)}{\partial\Pi^a(z)} = 0$, which reduces the above equation to

$$[\varepsilon_0(x), H] = \int dz \frac{\partial\varepsilon_0(x)}{\partial\varepsilon_a(z)}\frac{\partial H}{\partial\Pi^a(z)}. \tag{256}$$

By definition

$$\frac{\partial\varepsilon_0(x)}{\partial\varepsilon_a(z)} = \delta^0_a\delta(z-x), \tag{257}$$

and so we're only concerned with

$$\frac{\partial H}{\partial\Pi^0(z)} = \left[\left(\frac{1}{2(a_1+a_2+a_3)}\right)\Pi^0(z) + \left(\frac{a_2}{(a_1+a_2+a_3)}\right)(\partial_i\varepsilon_i(z))\right]. \tag{258}$$

Thus,

$$[\varepsilon_0(x), H] = \left(\frac{1}{2(a_1+a_2+a_3)}\right)\Pi^0(x) + \left(\frac{a_2}{(a_1+a_2+a_3)}\right)(\partial_i\varepsilon_i(x)). \tag{259}$$

Now we look at

$$\left[\varepsilon_0(x), \int dy\ u_1(y)\phi_1(y)\right] = \int dy\ u_1(y)\left[\varepsilon_0(x), \phi_1(y)\right]$$
$$= \int dz \int dy\ u_1(y)\left(\frac{\partial\varepsilon_0(x)}{\partial\varepsilon_a(z)}\frac{\partial\phi_1(y)}{\partial\Pi^a(z)}\right) \tag{260}$$
$$= \int dz \int dy\ u_1(y)\left[\delta^0_a\delta(z-x)\right]\left[\delta^\lambda_a\delta(z-y)\right] = \mathbf{0}.$$

From the equations above, we have thus shown that

$$\dot{\varepsilon_{\mathbf{0}}} = \left(\frac{\mathbf{1}}{\mathbf{2(a_1+a_2+a_3)}}\right)\mathbf{\Pi^0(x)} + \left(\frac{\mathbf{a_2}}{\mathbf{(a_1+a_2+a_3)}}\right)(\partial_{\mathbf{i}}\varepsilon_{\mathbf{i}}(\mathbf{x})). \tag{261}$$

Let's now look at

$$\dot{\varepsilon_i} = [\varepsilon_i, H_E] = [\varepsilon_i(x), H] + \left[\varepsilon_i(x), \int dy\ u_1(y)\phi_1(y)\right], \tag{262}$$

and start with

$$[\varepsilon_i(x), H] = \int dz\left(\frac{\partial\varepsilon_i(x)}{\partial\varepsilon_a(z)}\frac{\partial H}{\partial\Pi^a(z)} - \frac{\partial\varepsilon_i(x)}{\partial\Pi^a(z)}\frac{\partial H}{\partial\varepsilon_a(z)}\right). \tag{263}$$

which, since $\frac{\partial\varepsilon_i(x)}{\partial\Pi^a(z)} = 0$, can immediately be reduced to

$$[\varepsilon_i(x), H] = \int dz \frac{\partial\varepsilon_i(x)}{\partial\varepsilon_a(z)}\frac{\partial H}{\partial\Pi^a(z)}. \tag{264}$$

Now, by definition,

$$\frac{\partial\varepsilon_i(x)}{\partial\varepsilon_a(z)} = \delta^i_a\delta(z-x), \tag{265}$$

which means that we only have to concern ourselves with

$$\frac{\partial H}{\partial\Pi^i(z)} = \left[-\left(\frac{1}{2a_1}\right)\Pi^i(z) - \left(\frac{a_3}{a_1}\right)(\partial_i\varepsilon_0(z))\right]\delta^i_a. \tag{266}$$

Clearly, then,

$$[\varepsilon_i(x), H] = \left[-\left(\frac{1}{2a_1}\right)\Pi^i(x) - \left(\frac{a_3}{a_1}\right)(\partial_i\varepsilon_0(x))\right]. \tag{267}$$

Now we look at

$$\left[\varepsilon_i(x), \int dy\ u_1(y)\phi_1(y)\right] = \int dy\ u_1(y)\left[\varepsilon_i(x), \phi_1(y)\right]$$
$$= \int dz \int dy\ u_1(y)\left(\frac{\partial\varepsilon_i(x)}{\partial\varepsilon_a(z)}\frac{\partial\phi_1(y)}{\partial\Pi^a(z)}\right) \tag{268}$$
$$= \int dz \int dy\ u_1(y)\left[\delta^i_a\delta(z-x)\right]\left[\delta^\lambda_a\delta(z-y)\right] = \mathbf{0}.$$

From the equations above, we have thus effectively shown that

$$\dot{\varepsilon_{\mathbf{i}}} = -\left(\frac{\mathbf{1}}{\mathbf{2a_1}}\right)\mathbf{\Pi^i}(\mathbf{x}) - \left(\frac{\mathbf{a_3}}{\mathbf{a_1}}\right)(\partial_{\mathbf{i}}\varepsilon_{\mathbf{0}}(\mathbf{x})). \tag{269}$$

We finally look at

$$\dot{\varepsilon_\lambda} = [\varepsilon_\lambda, H_E] = [\varepsilon_\lambda(x), H] + \left[\varepsilon_\lambda(x), \int dy\ u_1(y)\phi_1(y)\right], \tag{270}$$

And again start with

$$[\varepsilon_\lambda(x), H] = \int dz \left(\frac{\partial\varepsilon_\lambda(x)}{\partial\varepsilon_a(z)}\frac{\partial H}{\partial\Pi^a(z)} - \frac{\partial\varepsilon_\lambda(x)}{\partial\Pi^a(z)}\frac{\partial H}{\partial\varepsilon_a(z)}\right). \tag{271}$$

Again, we can see readily that $\frac{\partial\varepsilon_\lambda(x)}{\partial\Pi^a(z)} = 0$, thus the above equation reduces to

$$[\varepsilon_\lambda(x), H] = \int dz \frac{\partial\varepsilon_\lambda(x)}{\partial\varepsilon_a(z)}\frac{\partial H}{\partial\Pi^a(z)}. \tag{272}$$

By definition,

$$\frac{\partial\varepsilon_\lambda(x)}{\partial\varepsilon_a(z)} = \delta_a^\lambda\delta(z-x), \tag{273}$$

which limits our concern to

$$\frac{\partial H}{\partial\Pi^\lambda(z)} = 0; \tag{274}$$

Thus,

$$[\varepsilon_\lambda(x), H] = \mathbf{0}. \tag{275}$$

Now we look at

$$\begin{aligned}\left[\varepsilon_\lambda(x), \int dy\ u_1(y)\phi_1(y)\right] &= \int dy\ u_1(y)\,[\varepsilon_\lambda(x), \phi_1(y)] \\ &= \int dz \int dy\ u_1(y)\left(\frac{\partial\varepsilon_\lambda(x)}{\partial\varepsilon_a(z)}\frac{\partial\phi_1(y)}{\partial\Pi^a(z)}\right) \\ &= \int dz \int dy\ u_1(y)\left[\delta_a^\lambda\delta(z-x)\right]\left[\delta_a^\lambda\delta(z-y)\right] \\ &= \mathbf{u_1}(\mathbf{x}).\end{aligned} \tag{276}$$

Thus, we have effectively shown that

$$\dot{\varepsilon_\lambda} = \mathbf{u_1}(\mathbf{x}), \tag{277}$$

with $u_1(x)$ as defined above in (245).

Now let's start to crack at the momenta equations of motion. We'll begin with

$$\dot{\Pi^0} = \left[\Pi^0(x), H_E\right] = \left[\Pi^0(x), H\right] + \left[\Pi^0(x), \int dy\ u_1(y)\phi_1(y)\right]. \tag{278}$$

We'll start with

$$\left[\Pi^0(x), H\right] = \int dz\ \frac{\partial\Pi^0(x)}{\partial\varepsilon_a(z)}\frac{\partial H}{\partial\Pi^a(z)} - \frac{\partial\Pi^0(x)}{\partial\Pi^a(z)}\frac{\partial H}{\partial\varepsilon_a(z)}, \tag{279}$$

but immediately we recognize that $\frac{\partial\Pi^0(x)}{\partial\varepsilon_a(z)} = 0$, and thus the above equation reduces to

$$\left[\Pi^0(x), H\right] = -\int dz\ \frac{\partial\Pi^0(x)}{\partial\Pi^a(z)}\frac{\partial H}{\partial\varepsilon_a(z)}. \tag{280}$$

By definition

$$\frac{\partial\Pi^0(x)}{\partial\Pi^a(z)} = \delta^0_a\delta(z-x), \tag{281}$$

and thus we're only interested in

$$\frac{\partial H}{\partial\varepsilon_0(z)} = -2\left(\frac{(a_1)^2-(a_3)^2}{a_1}\right)\partial_i\partial_i\varepsilon_0(z) + \left(\frac{a_3}{a_1}\right)\partial_i\Pi^i(z) + 2\lambda\varepsilon_0(z). \tag{282}$$

Hence,

$$\left[\Pi^0(x), H\right] = 2\left(\frac{(a_1)^2-(a_3)^2}{a_1}\right)\partial_i\partial_i\varepsilon_0(x) - \left(\frac{a_3}{a_1}\right)\partial_i\Pi^i(x) - 2\lambda\varepsilon_0(x). \tag{283}$$

Now we can look at

$$\begin{aligned}\left[\Pi^0(x), \int dy\ u_1(y)\phi_1(y)\right] &= \int dy\ u_1(y)\left[\Pi^0(x), \phi_1(y)\right] \\ &= -\int dz \int dy\ u_1(y)\frac{\partial\Pi^0(x)}{\partial\Pi^a(z)}\frac{\partial\phi_1(y)}{\partial\varepsilon_a(z)}.\end{aligned} \tag{284}$$

But recall that $\frac{\partial\phi_1(y)}{\partial\varepsilon_a(z)} = 0$, and thus we have that

$$\left[\Pi^0(x), \int dy\ u_1(y)\phi_1(y)\right] = \mathbf{0}. \tag{285}$$

From the information above, we have thus shown that

$$\boldsymbol{\dot{\Pi}^0 = 2\left(\frac{(a_1)^2-(a_3)^2}{a_1}\right)\partial_i\partial_i\varepsilon_0(x) - \left(\frac{a_3}{a_1}\right)\partial_i\Pi^i(x) - 2b\lambda}. \tag{286}$$

Let's turn now to investigating

$$\dot{\Pi}^i = \left[\Pi^i(x), H_E\right] = \left[\Pi^i(x), H\right] + \left[\Pi^i(x), \int dy\ u_1(y)\phi_1(y)\right]; \tag{287}$$

Again we'll begin with

$$\left[\Pi^i(x), H\right] = -\int dz\ \frac{\partial\Pi^i(x)}{\partial\Pi^a(z)}\frac{\partial H}{\partial\varepsilon_a(z)}. \tag{288}$$

By definition

$$\frac{\partial \Pi^i(x)}{\partial \Pi^a(z)} = \delta^i_a \delta(z - x), \tag{289}$$

and thus we're only interested in

$$\begin{aligned}\frac{\partial H}{\partial \varepsilon_i(z)} = {}& 2a_1(\partial_k \partial_k \varepsilon_i(z)) + 2\left(\left(\frac{-(a_2)^2}{(a_1 + a_2 + a_3)}\right) + a_2 + a_3\right)(\partial_i \partial_k \varepsilon_k(z)) \\ &- \left(\frac{a_2}{(a_1 + a_2 + a_3)}\right)\partial_i \Pi^0(z).\end{aligned} \tag{290}$$

Thus,

$$\begin{aligned}\left[\Pi^i(x), H\right] = {}& -2a_1(\partial_k \partial_k \varepsilon_i(x)) + 2\left(\left(\frac{(a_2)^2}{(a_1 + a_2 + a_3)}\right) - a_2 - a_3\right)(\partial_i \partial_k \varepsilon_k(x)) \\ &+ \left(\frac{a_2}{(a_1 + a_2 + a_3)}\right)\partial_i \Pi^0(x).\end{aligned} \tag{291}$$

Next we consider

$$\begin{aligned}\left[\Pi^i(x), \int dy\ u_1(y)\phi_1(y)\right] &= \int dy\ u_1(y)\left[\Pi^i(x), \phi_1(y)\right] \\ &= -\int dz \int dy\ u_1(y)\frac{\partial \Pi^i(x)}{\partial \Pi^a(z)}\frac{\partial \phi_1(y)}{\partial \varepsilon_a(z)}.\end{aligned} \tag{292}$$

But again, we know that $\frac{\partial \phi_1(y)}{\partial \varepsilon_a(z)} = 0$, and thus we have that

$$\left[\Pi^i(x), \int dy\ u_1(y)\phi_1(y)\right] = \mathbf{0}. \tag{293}$$

From the information above, we have effectively shown that

$$\boldsymbol{\begin{aligned}\dot{\Pi}^i = {}& -2a_1(\partial_k \partial_k \varepsilon_i(x)) + 2\left(\left(\frac{(a_2)^2}{(a_1 + a_2 + a_3)}\right) - a_2 - a_3\right)(\partial_i \partial_k \varepsilon_k(x)) \\ &+ \left(\frac{a_2}{(a_1 + a_2 + a_3)}\right)\partial_i \Pi^0(x).\end{aligned}} \tag{294}$$

Lastly, we turn our attention to

$$\dot{\Pi}^\lambda = \left[\Pi^\lambda(x), H_E\right] = \left[\Pi^\lambda(x), H\right] + \left[\Pi^\lambda(x), \int dy\ u_1(y)\phi_1(y)\right]. \tag{295}$$

Starting with

$$\left[\Pi^\lambda(x), H\right] = -\int dz\ \frac{\partial \Pi^\lambda(x)}{\partial \Pi^a(z)}\frac{\partial H}{\partial \varepsilon_a(z)}, \tag{296}$$

we know that

$$\frac{\partial \Pi^\lambda(x)}{\partial \Pi^a(z)} = \delta^\lambda_a \delta(z - x) \tag{297}$$

by definition, and thus we're only concerned with

$$\frac{\partial H}{\partial \varepsilon_\lambda(z)} = -2b\varepsilon_0(z). \tag{298}$$

It is thus apparent that

$$\left[\Pi^\lambda(x), H\right] = -2b\varepsilon_0(x). \tag{299}$$

Now we can look at

$$\begin{aligned}\left[\Pi^\lambda(x), \int dy\ u_1(y)\phi_1(y)\right] &= \int dy\ u_1(y)\left[\Pi^\lambda(x), \phi_1(y)\right] \\ &= -\int dz \int dy\ u_1(y)\frac{\partial \Pi^\lambda(x)}{\partial \Pi^a(z)}\frac{\partial \phi_1(y)}{\partial \varepsilon_a(z)}.\end{aligned} \tag{300}$$

But, just as in the two previous momenta derivations, we know that $\frac{\partial \phi_1(y)}{\partial \varepsilon_a(z)} = 0$, and thus we have again that

$$\left[\Pi^i(x), \int dy\ u_1(y)\phi_1(y)\right] = \mathbf{0}. \tag{301}$$

And thus, pulling all of this together, we have the final momentum equation of motion:

$$\dot{\mathbf{\Pi}}^\lambda = -\mathbf{2b}\varepsilon_\mathbf{0}(\mathbf{x}). \tag{302}$$

Now, from the classifications of the constraints in (252), we know that this vector theory has

$$\mathcal{N} = 10\ Field\ Degrees\ of\ Freedom \tag{303}$$
$$n_1 = 0\ First - Class\ Constraints \tag{304}$$
$$n_2 = 4\ Second - Class\ Constraints \tag{305}$$

Which means that it still has

$$\mathcal{N} - 2n_1 - n_2 = 10 - 2(0) - 4 = \ \mathbf{6}\ unaccounted\, degrees\, of\ freedom. \tag{306}$$

Interestingly, in this linearized case–as in the non-linearized case considered above–we still have six degrees of freedom. However, in this case Eq.(209) gives $\varepsilon_0 = 0$ directly, leaving only the (six) ε_j and Π^j components for consideration. These are the Nambu-Goldstone modes (two transverse modes, each with conjugate momenta) and potentially a massive mode as well; again this could depend on choices of a_1, a_2, and a_3. In any case, the results of the linearized approximation of the Lagrange-Multiplier potential do not directly reduce to those of Electromagnetism, as is evidenced in the different constraint structure and number of degrees of freedom between the two theories.

5.4 Vector Theory with a Constant and a Square

5.4.1 Overview

In this analysis, we simply use a smooth quadratic potential,

$$V = \frac{1}{2}\kappa(A_\mu A^\mu \pm b^2)^2, \qquad \kappa = const. \tag{307}$$

introducing no extra fields as in the case with the Lagrange-Multiplier Field potential. In this manner, the Poisson Bracket resumes its familiar form in (15).

5.4.2 Analysis

We start with the Lagrangian Density

$$\begin{aligned}\mathcal{L} = &-a_1(\partial_i A_0)^2 - a_1(\partial_0 A_i)^2 + a_1(\partial_i A_j)^2 + a_2(\partial_i A_i)(\partial_j A_j) - 2a_2(\partial_0 A_0)(\partial_i A_i)\\ &+ a_3(\partial_i A_j)(\partial_j A_i) - 2a_3(\partial_i A_0)(\partial_0 A_i) + (a_1+a_2+a_3)(\partial_0 A_0)^2 - \frac{1}{2}\kappa((A_0)^2 - (A_i)^2 \pm b^2)^2,\end{aligned} \tag{308}$$

and from it we calculate the Conjugate Momenta

$$\Pi^0 = 2(a_1+a_2+a_3)(\partial_0 A_0) - 2a_2(\partial_i A_i) \tag{309}$$

$$\Pi^i = -2a_1(\partial_0 A_i) - 2a_3(\partial_i A_0) \tag{310}$$

Now, interestingly here we don't find *ANY* primary constraints, and so we simply use the conjugate momenta to calculate the Hamiltonian Density:

$$\begin{aligned}\mathcal{H} = &\left(\frac{(a_1)^2-(a_3)^2}{a_1}\right)(\partial_i A_0)^2 - \left(\frac{1}{4a_1}\right)(\Pi^i)^2 - \left(\frac{a_3}{a_1}\right)\Pi^i\partial_i A_0 - a_1(\partial_i A_j)^2\\ &- a_2(\partial_i A_i)(\partial_j A_j) - a_3(\partial_i A_j)(\partial_j A_i) + \left(\frac{1}{4(a_1+a_2+a_3)}\right)(\Pi^0)^2\\ &+ \left(\frac{a_2}{(a_1+a_2+a_3)}\right)\Pi^0\partial_i A_i + \left(\frac{(a_2)^2}{(a_1+a_2+a_3)}\right)(\partial_i A_i)^2 + \frac{1}{2}\kappa((A_0)^2-(A_i)^2\pm b^2)^2.\end{aligned} \tag{311}$$

Also, without any constraints, we know that

$$H_E = H_T = H = \int dx\mathcal{H}(x), \tag{312}$$

and we can begin to calculate the equations of motion. We'll start with the fields...

Starting with

$$\dot{A_0} = [A_0(x), H] = \int dz\left(\frac{\partial A_0(x)}{\partial A_\mu(z)}\frac{\partial H}{\partial\Pi^\mu(z)} - \frac{\partial A_0(x)}{\partial\Pi^\mu(z)}\frac{\partial H}{\partial A_\mu(z)}\right), \tag{313}$$

we immediately recognize that $\frac{\partial A_0(x)}{\partial\Pi^\mu(z)} = 0$, so we can simplify

$$[A_0(x), H] = \int dz\left(\frac{\partial A_0(x)}{\partial A_\mu(z)}\frac{\partial H}{\partial\Pi^\mu(z)}\right). \tag{314}$$

By definition

$$\frac{\partial A_0(x)}{\partial A_\mu(z)} = \delta^0_\mu \delta(z-x), \tag{315}$$

and thus we're only considering

$$\frac{\partial H}{\partial \Pi^0(z)} = \left[\left(\frac{1}{2(a_1+a_2+a_3)}\right)\Pi^0(z) + \left(\frac{a_2}{(a_1+a_2+a_3)}\right)(\partial_i A_i(z))\right]. \tag{316}$$

So, we find that

$$\mathbf{\dot{A_0}} = \left[\left(\frac{\mathbf{1}}{\mathbf{2(a_1+a_2+a_3)}}\right)\mathbf{\Pi^0(x)} + \left(\frac{\mathbf{a_2}}{\mathbf{(a_1+a_2+a_3)}}\right)(\partial_\mathbf{i}\mathbf{A_i(x)})\right]. \tag{317}$$

Next we look at

$$\dot{A}_i = [A_i(x), H] = \int dz \left(\frac{\partial A_i(x)}{\partial A_\mu(z)}\frac{\partial H}{\partial \Pi^\mu(z)}\right). \tag{318}$$

By definition,

$$\frac{\partial A_i(x)}{\partial A_\mu(z)} = \delta^i_\mu \delta(z-x), \tag{319}$$

and thus we are only concerned with

$$\frac{\partial H}{\partial \Pi^i(z)} = -\left(\frac{1}{2a_1}\right)\Pi^i(z) - \left(\frac{a_3}{a_1}\right)(\partial_i A_0(z)); \tag{320}$$

It is straightforward, then, that

$$\mathbf{\dot{A}_i} = -\left(\frac{\mathbf{1}}{\mathbf{2a_1}}\right)\mathbf{\Pi^i(x)} - \left(\frac{\mathbf{a_3}}{\mathbf{a_1}}\right)(\partial_\mathbf{i}\mathbf{A_0(x)}). \tag{321}$$

Looking at the momenta equations of motion, we first consider

$$\dot{\Pi^0} = \left[\Pi^0(x), H\right] = \int dz\ \frac{\partial \Pi^0(x)}{\partial A_\mu(z)}\frac{\partial H}{\partial \Pi^\mu(z)} - \frac{\partial \Pi^0(x)}{\partial \Pi^\mu(z)}\frac{\partial H}{\partial A_\mu(z)}, \tag{322}$$

but immediately we recall that $\frac{\partial \Pi^0(x)}{\partial A_\mu(z)} = 0$, and thus the above equation reduces to

$$\left[\Pi^0(x), H\right] = -\int dz\ \frac{\partial \Pi^0(x)}{\partial \Pi^\mu(z)}\frac{\partial H}{\partial A_\mu(z)}. \tag{323}$$

By definition,

$$\frac{\partial \Pi^0(x)}{\partial \Pi^\mu(z)} = \delta^0_\mu \delta(z-x), \tag{324}$$

thus we can limit our focus to

$$\begin{aligned}\frac{\partial H}{\partial A_0(z)} &= -2\left(\frac{(a_1)^2-(a_3)^2}{a_1}\right)\partial_i\partial_i A_0(z) + \left(\frac{a_3}{a_1}\right)\partial_i \Pi^i(z) \\ &\quad + 2\kappa A_0(z)\left((A_0(z))^2 - (A_i(z))^2 \pm (b)^2\right).\end{aligned} \tag{325}$$

Thus we see that

$$\boldsymbol{\dot{\Pi}^0 = 2\left(\frac{(a_1)^2-(a_3)^2}{a_1}\right)\partial_i\partial_i A_0(x) - \left(\frac{a_3}{a_1}\right)\partial_i\Pi^i(x)} \\ \boldsymbol{- 2\kappa A_0(x)\left((A_0(x))^2-(A_i(x))^2 \pm (b)^2\right).} \tag{326}$$

Lastly we consider

$$\dot{\Pi}^i = \left[\Pi^i(x), H\right] = -\int dz\ \frac{\partial \Pi^i(x)}{\partial \Pi^\mu(z)}\frac{\partial H}{\partial A_\mu(z)}; \tag{327}$$

By definition

$$\frac{\partial \Pi^i(x)}{\partial \Pi^\mu(z)} = \delta^i_\mu \delta(z-x), \tag{328}$$

and thus we only need to look at

$$\frac{\partial H}{\partial A_i(z)} = 2a_1(\partial_k\partial_k A_i(z)) + 2\left(-\left(\frac{(a_2)^2}{(a_1+a_2+a_3)}\right)+a_2+a_3\right)(\partial_i\partial_k A_k(z)) \\ -\left(\frac{a_2}{(a_1+a_2+a_3)}\right)\partial_i\Pi^0(z) - 2\kappa A_i(z)\left((A_0(z))^2-(A_i(z))^2 \pm (b)^2\right). \tag{329}$$

Hence we have the final momentum equation of motion:

$$\boldsymbol{\dot{\Pi}^i = -2a_1(\partial_k\partial_k A_i(z)) + 2\left(\left(\frac{(a_2)^2}{(a_1+a_2+a_3)}\right)-a_2-a_3\right)(\partial_i\partial_k A_k(z))} \\ \boldsymbol{+\left(\frac{a_2}{(a_1+a_2+a_3)}\right)\partial_i\Pi^0(z) + 2\kappa A_i(z)\left((A_0(z))^2-(A_i(z))^2 \pm (b)^2\right).} \tag{330}$$

With the information above, we understand that this vector theory has

$$\mathcal{N} = 8\ Field\,Degrees\,of\,Freedom \tag{331}$$
$$n_1 = 0\ First - Class\,Constraints \tag{332}$$
$$n_2 = 0\ Second - Class\,Constraints \tag{333}$$

Which means that it still has

$$\mathcal{N} - 2n_1 - n_2 = 8 - 2(0) - 0 = \ \mathbf{8}\ unaccounted\,degrees\,of\,freedom. \tag{334}$$

Immediately this case proves interesting due to the fact that it is entirely unconstrained; from the discrepancy in the number of degrees of freedom, this theory proves starkly different than E&M. Additionally, since none of the field components in this theory are constrained, the potential presence of ghost modes in this particular bumblebee model could prove problematic if all four A_μ do in fact propagate.

5.5 Vector Theory with a Lagrange-Multiplier Field and a Square

5.5.1 Overview

Here we return to the consideration of a potential with a Lagrange-Multiplier Field, this time with the quadratic form of the potential:

$$V = \frac{1}{2}\lambda(A_\mu A^\mu \pm b^2)^2. \tag{335}$$

Again we accrue the extra dynamical field λ (which is introduced simply to impose a constraint, but then drops out of the equations of motion) and thus the Poisson Bracket now returns to its definition in (69).

5.5.2 Analysis

Here we have the Lagrangian

$$\begin{aligned}\mathcal{L} = &-a_1(\partial_i A_0)^2 - a_1(\partial_0 A_i)^2 + a_1(\partial_i A_j)^2 + a_2(\partial_i A_i)(\partial_j A_j) - 2a_2(\partial_0 A_0)(\partial_i A_i)\\ &+ a_3(\partial_i A_j)(\partial_j A_i) - 2a_3(\partial_i A_0)(\partial_0 A_i) + (a_1+a_2+a_3)(\partial_0 A_0)^2 - \frac{1}{2}\lambda((A_0)^2 - (A_i)^2 \pm b^2)^2,\end{aligned} \tag{336}$$

which gives the following conjugate momenta

$$\Pi^0 = 2(a_1+a_2+a_3)(\partial_0 A_0) - 2a_2(\partial_i A_i) \tag{337}$$

$$\Pi^i = -2a_1(\partial_0 A_i) - 2a_3(\partial_i A_0) \tag{338}$$

$$\Pi^\lambda = 0. \tag{339}$$

Here again we immediately recognize and record the Primary Constraint

$$\phi_1 = \Pi^\lambda, \tag{340}$$

but we break for a second to use our conjugate momenta to calculate the Hamiltonian:

$$\begin{aligned}\mathcal{H} = &\left(\frac{(a_1)^2-(a_3)^2}{a_1}\right)(\partial_i A_0)^2 - \left(\frac{1}{4a_1}\right)(\Pi^i)^2 - \left(\frac{a_3}{a_1}\right)\Pi^i\partial_i A_0 - a_1(\partial_i A_j)^2\\ &- a_2(\partial_i A_i)(\partial_j A_j) - a_3(\partial_i A_j)(\partial_j A_i) + \left(\frac{1}{4(a_1+a_2+a_3)}\right)(\Pi^0)^2\\ &+ \left(\frac{a_2}{(a_1+a_2+a_3)}\right)\Pi^0\partial_i A_i + \left(\frac{(a_2)^2}{(a_1+a_2+a_3)}\right)(\partial_i A_i)^2 + \frac{1}{2}\lambda((A_0)^2-(A_i)^2 \pm b^2)^2.\end{aligned} \tag{341}$$

Now we use this Hamiltonian again to help calculate the Time-Evolution of ϕ_1,

$$\dot{\phi_1} = [\phi_1(x), H] + \frac{\partial \phi_1}{\partial t}. \tag{342}$$

Clearly, again, $\frac{\partial \phi_1}{\partial t} = 0$, so the above equation reduces to:

$$\dot{\phi_1} = [\phi_1(x), H] = \int dz \left(\frac{\partial \phi_1(x)}{\partial A_a(z)}\frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial \phi_1(x)}{\partial \Pi^a(z)}\frac{\partial H}{\partial A_a(z)}\right) \tag{343}$$

But $\frac{\partial\phi_1(x)}{\partial A_a(z)} = 0$, so again the above equation is simply

$$\dot{\phi_1} = [\phi_1(x), H] = -\int dz \frac{\partial\phi_1(x)}{\partial\Pi^a(z)} \frac{\partial H}{\partial A_a(z)}. \tag{344}$$

Now, by definition,

$$\frac{\partial\phi_1(x)}{\partial\Pi^a(z)} = \delta_a^\lambda \delta(z-x) \tag{345}$$

and thus we only need to worry about

$$\frac{\partial H}{\partial A_\lambda(z)} = \frac{1}{2}\left((A_0(z))^2 - (A_i(z))^2 \pm (b)^2\right)^2. \tag{346}$$

Thus, from the above equations, we see that

$$\dot{\phi_1} = -\frac{1}{2}\left((A_0(x))^2 - (A_i(x))^2 \pm (b)^2\right)^2 \tag{347}$$

And we require this to be equal to zero, to keep the time-evolution of constraints constant. Thus we have defined a new, secondary constraint,

$$\phi_2 = -\frac{1}{2}((A_0)^2 - (A_i)^2 \pm b^2)^2. \tag{348}$$

Quite clearly, ϕ_2 above is a *nonlinear constraint*, as discussed in **Section 4.3.2**. For the rest of this analysis, we will use the assumption that (31) holds, which is to say that the linear approximation

$$\chi_2 = ((A_0)^2 - (A_i)^2 \pm b^2) \tag{349}$$

is second-class, and indeed equivalent to its nonlinear counterpart, the constraint ϕ_2. We will show that this leads to a constraint structure and numeration of degrees of freedom according to Dirac's counting argument which matches *exactly* those determined by the Lagrangian approach.

An alternative assumption would be that the linear approximation to ϕ_2 above in (349) is in fact *first-class* and thus (31) *does not hold*, but in such a scenario it can be shown that the constraint evolution does not truncate, and quickly the system becomes overconstrained according to Dirac's counting argument. The results of such a procedure are thus unphysical, and point to the failure of the Dirac method in various irregular cases as discussed above in **Section 4.3**.

So, we continue with the investigation of the time-evolution of this new secondary constraint ϕ_2, using that

$$\dot{\phi_2} = [\phi_2(x), H_T] + \frac{\partial\phi_2}{\partial t} = [\phi_2(x), H] + \left[\phi_2(x), \int u_1(y)\phi_1(y)dy\right] + \frac{\partial\phi_2}{\partial t}. \tag{350}$$

Immediately we see that $\frac{\partial\phi_2}{\partial t} = 0$, so the above equation reduces to

$$\dot{\phi_2} = [\phi_2(x), H_T] = [\phi_2(x), H] + \left[\phi_2(x), \int u_1(y)\phi_1(y)dy\right]. \tag{351}$$

Investigating

$$[\phi_2(x), H] = \int dz \left(\frac{\partial \phi_2(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial \phi_2(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial A_a(z)} \right) \tag{352}$$

we recognize that $\frac{\partial \phi_2(x)}{\partial \Pi^a(z)} = 0$, which again reduces the above equation to

$$[\phi_2(x), H] = \int dz \frac{\partial \phi_2(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)}. \tag{353}$$

Now,

$$\frac{\partial \phi_2(x)}{\partial A_a(z)} = \left(-2A_0\delta_a^0 + 2A_i\delta_a^i\right)\left((A_0)^2 - (A_i)^2 \pm b^2\right)\delta(z-x) \tag{354}$$

and

$$\begin{aligned} \frac{\partial H}{\partial \Pi^a(z)} = &\left[-\left(\frac{1}{2a_1}\right)\Pi^i - \left(\frac{a_3}{a_1}\right)(\partial_i A_0)\right]\delta_a^i \\ &+ \left[\left(\frac{1}{2(a_1+a_2+a_3)}\right)\Pi^0 + \left(\frac{a_2}{(a_1+a_2+a_3)}\right)(\partial_i A_i)\right]\delta_a^0. \end{aligned} \tag{355}$$

Thus we find that, after some finesse,

$$\begin{aligned} [\phi_2(x), H] = &\left[-\left(\frac{1}{a_1}\right)A_i\Pi^i - 2\left(\frac{a_3}{a_1}\right)A_i\partial_i A_0 - \left(\frac{1}{(a_1+a_2+a_3)}\right)A_0\Pi^0 \right. \\ &\left. -2\left(\frac{a_2}{(a_1+a_2+a_3)}\right)\partial_i A_i\right]\left((A_0(x))^2 - (A_i(x))^2 \pm (b)^2\right). \end{aligned} \tag{356}$$

Now, investigating

$$\begin{aligned} \left[\phi_2(x), \int u_1(y)\phi_1(y)dy\right] &= \int dy\ u_1(y)\left[\phi_2(x), \phi_1(y)\right] \\ &= \int dz \int dy\ u_1(y)\left(\frac{\partial \phi_2(x)}{\partial A_a(z)}\frac{\partial \phi_1(y)}{\partial \Pi^a(z)} - \frac{\partial \phi_2(x)}{\partial \Pi^a(z)}\frac{\partial \phi_1(y)}{\partial A_a(z)}\right) \end{aligned} \tag{357}$$

We first recall that $\frac{\partial \phi_1(x)}{\partial A_a(z)} = 0$, which reduces the above equation to

$$\left[\phi_2(x), \int u_1(y)\phi_1(y)dy\right] = \int dz \int dy\ u_1(y)\left(\frac{\partial \phi_2(x)}{\partial A_a(z)}\frac{\partial \phi_1(y)}{\partial \Pi^a(z)}\right). \tag{358}$$

And we see quite readily that

$$\left(\frac{\partial \phi_2(x)}{\partial A_a(z)}\frac{\partial \phi_1(y)}{\partial \Pi^a(z)}\right) = \left[\left((A_0)^2 - (A_i)^2 \pm b^2\right)\left(-2A_0\delta_a^0 + 2A_i\delta_a^i\right)\delta(z-x)\right]\left[\delta_a^\lambda\delta(z-y)\right] = 0. \tag{359}$$

Thus, from the information above, we know that

$$\begin{aligned} \dot{\phi}_2 = &\left[-\left(\frac{1}{a_1}\right)A_i\Pi^i - 2\left(\frac{a_3}{a_1}\right)A_i\partial_i A_0 - \left(\frac{1}{(a_1+a_2+a_3)}\right)A_0\Pi^0 \right. \\ &\left. -2\left(\frac{a_2}{(a_1+a_2+a_3)}\right)\partial_i A_i\right]\left((A_0(x))^2 - (A_i(x))^2 \pm (b)^2\right). \end{aligned} \tag{360}$$

Again, we require that the time-evolution of this constraint be constant, and so we set the above equation to zero. This simplifies to

$$\begin{aligned}&\left[\left(\frac{1}{a_1}\right) A_i \Pi^i + 2\left(\frac{a_3}{a_1}\right) A_i \partial_i A_0 + \left(\frac{1}{(a_1+a_2+a_3)}\right) A_0 \Pi^0\right.\\ &\left.+2\left(\frac{a_2}{(a_1+a_2+a_3)}\right)\partial_i A_i\right]\left((A_0)^2 - (A_i)^2 \pm b^2\right) = 0.\end{aligned} \tag{361}$$

Having agreed on the validity of (31) in the analysis of this particular bumblebee model, we are free to say that

$$\phi_2 \approx 0 \Rightarrow \chi_2 \approx 0. \tag{362}$$

Thus

$$\left(\left((A_0(x))^2 - (A_i(x))^2 \pm (b)^2\right)\right)^2 \approx 0 \Rightarrow \left((A_0(x))^2 - (A_i(x))^2 \pm (b)^2\right) \approx 0, \tag{363}$$

and (360) can be reduced simply to

$$\dot{\phi_2} = 0 \tag{364}$$

which is trivial, since we require this to be true of the time-evolution of any constraint. Thus no new constraints are bred from (360), and we begin with the classification of the (two) constraints in our system:

$$[\phi_1, \phi_2] = 0. \tag{365}$$

Thus we see that

$$\phi_1 = \mathbf{Primary},\ \mathbf{First-Class\ Constraint} \tag{366}$$
$$\phi_2 = \mathbf{Secondary},\ \mathbf{First-Class\ Constraint}, \tag{367}$$

and we can construct our Extended Hamiltonian

$$H_E = \int dx\ \mathcal{H}(x) + \int dx\ v_1(x)\phi_1(x) + \int dx\ v_2(x) \tag{368}$$

and use it to calculate the field and momenta equations of motion.

Starting with the field equations of motion, we consider

$$\dot{A_0} = [A_0, H_E] = [A_0(x), H] + \left[A_0(x), \int dy\ v_1(y)\phi_1(y)\right] + \left[A_0(x), \int dy\ v_2(y)\phi_2(y)\right], \tag{369}$$

and first investigate

$$[A_0(x), H] = \int dz \left(\frac{\partial A_0(x)}{\partial A_a(z)}\frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial A_0(x)}{\partial \Pi^a(z)}\frac{\partial H}{\partial A_a(z)}\right). \tag{370}$$

Immediately we see that $\frac{\partial A_0(x)}{\partial \Pi^a(z)} = 0$, which reduces the above equation to

$$[A_0(x), H] = \int dz \frac{\partial A_0(x)}{\partial A_a(z)}\frac{\partial H}{\partial \Pi^a(z)}. \tag{371}$$

By definition

$$\frac{\partial A_0(x)}{\partial A_a(z)} = \delta^0_a \delta(z-x), \tag{372}$$

and so we're only concerned with

$$\frac{\partial H}{\partial \Pi^0(z)} = \left[\left(\frac{1}{2(a_1+a_2+a_3)}\right)\Pi^0(z) + \left(\frac{a_2}{(a_1+a_2+a_3)}\right)(\partial_i A_i(z))\right]. \tag{373}$$

Thus,

$$[A_0(x), H] = \left(\frac{1}{2(a_1+a_2+a_3)}\right)\Pi^0(x) + \left(\frac{a_2}{(a_1+a_2+a_3)}\right)(\partial_i A_i(x)). \tag{374}$$

Now we look at

$$\begin{aligned}\left[A_0(x), \int dy\ v_1(y)\phi_1(y)\right] &= \int dy\ v_1(y)\,[A_0(x), \phi_1(y)] \\ &= \int dz \int dy\ v_1(y)\left(\frac{\partial A_0(x)}{\partial A_a(z)}\frac{\partial \phi_1(y)}{\partial \Pi^a(z)}\right) \\ &= \int dz \int dy\ v_1(y)\left[\delta^0_a\delta(z-x)\right]\left[\delta^\lambda_a\delta(z-y)\right] \\ &= \mathbf{0}.\end{aligned} \tag{375}$$

Finally we consider

$$\begin{aligned}\left[A_0(x), \int dy\ v_2(y)\phi_2(y)\right] &= \int dy\ v_2(y)\,[A_0(x), \phi_2(y)] \\ &= \int dz \int dy\ v_2(y)\left(\frac{\partial A_0(x)}{\partial A_a(z)}\frac{\partial \phi_2(y)}{\partial \Pi^a(z)}\right),\end{aligned} \tag{376}$$

but clearly

$$\frac{\partial \phi_2(y)}{\partial \Pi^a(z)} = 0 \tag{377}$$

and thus

$$\left[A_0(x), \int dy\ v_2(y)\phi_2(y)\right] = \mathbf{0}. \tag{378}$$

From the information above, we can see that

$$\dot{\mathbf{A}}_\mathbf{0} = \left(\frac{\mathbf{1}}{\mathbf{2(a_1+a_2+a_3)}}\right)\boldsymbol{\Pi}^\mathbf{0}(\mathbf{x}) + \left(\frac{\mathbf{a_2}}{(\mathbf{a_1+a_2+a_3})}\right)(\partial_\mathbf{i}\mathbf{A_i}(\mathbf{x})). \tag{379}$$

Let's now look at

$$\dot{A}_i = [A_i, H_E] = [A_i(x), H] + \left[A_i(x), \int dy\ v_1(y)\phi_1(y)\right] + \left[A_i(x), \int dy\ v_2(y)\phi_2(y)\right], \tag{380}$$

and start with

$$[A_i(x), H] = \int dz \left(\frac{\partial A_i(x)}{\partial A_a(z)}\frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial A_i(x)}{\partial \Pi^a(z)}\frac{\partial H}{\partial A_a(z)}\right), \tag{381}$$

which, since $\frac{\partial A_i(x)}{\partial \Pi^a(z)} = 0$, can immediately be reduced to

$$[A_i(x), H] = \int dz \frac{\partial A_i(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)}. \tag{382}$$

Now, by definition,

$$\frac{\partial A_i(x)}{\partial A_a(z)} = \delta^i_a \delta(z-x), \tag{383}$$

which means that we only have to concern ourselves with

$$\frac{\partial H}{\partial \Pi^i(z)} = \left[-\left(\frac{1}{2a_1}\right)\Pi^i(z) - \left(\frac{a_3}{a_1}\right)(\partial_i A_0(z)) \right] \delta^i_a. \tag{384}$$

Clearly, then,

$$[A_i(x), H] = \left[-\left(\frac{1}{2a_1}\right)\Pi^i(x) - \left(\frac{a_3}{a_1}\right)(\partial_i A_0(x)) \right]. \tag{385}$$

Now we look at

$$\begin{aligned} \left[A_i(x), \int dy\ v_1(y)\phi_1(y)\right] &= \int dy\ v_1(y)\left[A_i(x), \phi_1(y)\right] \\ &= \int dz \int dy\ v_1(y) \left(\frac{\partial A_i(x)}{\partial A_a(z)} \frac{\partial \phi_1(y)}{\partial \Pi^a(z)} \right) \\ &= \int dz \int dy\ v_1(y) \left[\delta^i_a \delta(z-x)\right] \left[\delta^\lambda_a \delta(z-y)\right] = \mathbf{0}. \end{aligned} \tag{386}$$

Finally we consider

$$\begin{aligned} \left[A_i(x), \int dy\ v_2(y)\phi_2(y)\right] &= \int dy\ v_2(y)\left[A_i(x), \phi_2(y)\right] \\ &= \int dz \int dy\ v_2(y) \left(\frac{\partial A_i(x)}{\partial A_a(z)} \frac{\partial \phi_2(y)}{\partial \Pi^a(z)} \right), \end{aligned} \tag{387}$$

but again, clearly

$$\frac{\partial \phi_2(y)}{\partial \Pi^a(z)} = 0 \tag{388}$$

and thus

$$\left[A_i(x), \int dy\ v_2(y)\phi_2(y)\right] = \mathbf{0}. \tag{389}$$

From the equations above, we have thus effectively shown that

$$\mathbf{\dot{A}_i} = \left[-\left(\mathbf{\frac{1}{2a_1}}\right)\mathbf{\Pi^i(x)} - \left(\mathbf{\frac{a_3}{a_1}}\right)(\mathbf{\partial_i A_0(x)}) \right]. \tag{390}$$

We finally look at

$$\dot{A_\lambda} = [A_\lambda, H_E] = [A_\lambda(x), H] + \left[A_\lambda(x), \int dy\ v_1(y)\phi_1(y)\right] + \left[A_\lambda(x), \int dy\ v_2(y)\phi_2(y)\right], \tag{391}$$

And again start with

$$[A_\lambda(x), H] = \int dz \left(\frac{\partial A_\lambda(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial A_\lambda(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial A_a(z)} \right). \tag{392}$$

Again, we can see readily that $\frac{\partial A_\lambda(x)}{\partial \Pi^a(z)} = 0$, thus the above equation reduces to

$$[A_\lambda(x), H] = \int dz \frac{\partial A_\lambda(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)}. \tag{393}$$

By definition,

$$\frac{\partial A_\lambda(x)}{\partial A_a(z)} = \delta_a^\lambda \delta(z - x), \tag{394}$$

which limits our concern to

$$\frac{\partial H}{\partial \Pi^\lambda(z)} = 0; \tag{395}$$

Thus,

$$[A_\lambda(x), H] = \mathbf{0}. \tag{396}$$

Now we look at

$$\begin{aligned} \left[A_\lambda(x), \int dy\ v_1(y)\phi_1(y) \right] &= \int dy\ v_1(y) \left[A_\lambda(x), \phi_1(y) \right] \\ &= \int dz \int dy\ v_1(y) \left(\frac{\partial A_\lambda(x)}{\partial A_a(z)} \frac{\partial \phi_1(y)}{\partial \Pi^a(z)} \right) \\ &= \int dz \int dy\ v_1(y) \left[\delta_a^\lambda \delta(z - x) \right] \left[\delta_a^\lambda \delta(z - y) \right] \\ &= \mathbf{v_1}(\mathbf{x}). \end{aligned} \tag{397}$$

Finally we consider

$$\begin{aligned} \left[A_\lambda(x), \int dy\ v_2(y)\phi_2(y) \right] &= \int dy\ v_2(y) \left[A_\lambda(x), \phi_2(y) \right] \\ &= \int dz \int dy\ v_2(y) \left(\frac{\partial A_\lambda(x)}{\partial A_a(z)} \frac{\partial \phi_2(y)}{\partial \Pi^a(z)} \right), \end{aligned} \tag{398}$$

but again, clearly

$$\frac{\partial \phi_2(y)}{\partial \Pi^a(z)} = 0 \tag{399}$$

and thus

$$\left[A_\lambda(x), \int dy\ v_2(y)\phi_2(y) \right] = \mathbf{0}. \tag{400}$$

Thus, we have effectively shown that

$$\dot{\mathbf{A}}_\lambda = \mathbf{v_1}(\mathbf{x}). \tag{401}$$

Now let's look at the momenta equations of motion. We start with

$$
\begin{aligned}
\dot{\Pi}^0 &= \left[\Pi^0(x), H_E\right] \\
&= \left[\Pi^0(x), H\right] + \left[\Pi^0(x), \int dy\ v_1(y)\phi_1(y)\right] + \left[\Pi^0(x), \int dy\ v_2(y)\phi_2(y)\right].
\end{aligned} \tag{402}
$$

We'll start with

$$
\left[\Pi^0(x), H\right] = \int dz\ \frac{\partial \Pi^0(x)}{\partial A_a(z)} \frac{\partial H}{\partial \Pi^a(z)} - \frac{\partial \Pi^0(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial A_a(z)}, \tag{403}
$$

but immediately we recognize that $\frac{\partial \Pi^0(x)}{\partial A_a(z)} = 0$, and thus the above equation reduces to

$$
\left[\Pi^0(x), H\right] = -\int dz\ \frac{\partial \Pi^0(x)}{\partial \Pi^a(z)} \frac{\partial H}{\partial A_a(z)}. \tag{404}
$$

By definition

$$
\frac{\partial \Pi^0(x)}{\partial \Pi^a(z)} = \delta^0_a \delta(z-x), \tag{405}
$$

and thus we're only interested in

$$
\begin{aligned}
\frac{\partial H}{\partial A_0(z)} = &-2\left(\frac{(a_1)^2 - (a_3)^2}{a_1}\right)\partial_i\partial_i A_0(z) + \left(\frac{a_3}{a_1}\right)\partial_i \Pi^i(z) \\
&+ 2\lambda A_0(z)\left((A_0(z))^2 - (A_i(z))^2 \pm (b)^2\right).
\end{aligned} \tag{406}
$$

Hence,

$$
\begin{aligned}
\left[\Pi^0(x), H\right] = &\ 2\left(\frac{(a_1)^2 - (a_3)^2}{a_1}\right)\partial_i\partial_i A_0(x) - \left(\frac{a_3}{a_1}\right)\partial_i \Pi^i(x) \\
&- 2\lambda A_0(x)\left((A_0(x))^2 - (A_i(x))^2 \pm (b)^2\right).
\end{aligned} \tag{407}
$$

Now we can look at

$$
\begin{aligned}
\left[\Pi^0(x), \int dy\ v_1(y)\phi_1(y)\right] &= \int dy\ v_1(y)\left[\Pi^0(x), \phi_1(y)\right] \\
&= -\int dz \int dy\ v_1(y) \frac{\partial \Pi^0(x)}{\partial \Pi^a(z)} \frac{\partial \phi_1(y)}{\partial A_a(z)}.
\end{aligned} \tag{408}
$$

But recall that $\frac{\partial \phi_1(y)}{\partial A_a(z)} = 0$, and thus we have that

$$
\left[\Pi^0(x), \int dy\ v_1(y)\phi_1(y)\right] = \mathbf{0}. \tag{409}
$$

Finally we consider

$$
\begin{aligned}
\left[\Pi^0(x), \int dy\ v_2(y)\phi_2(y)\right] &= \int dy\ v_2(y)\left[\Pi^0(x), \phi_2(y)\right] \\
&= -\int dz \int dy\ v_2(y) \frac{\partial \Pi^0(x)}{\partial \Pi^a(z)} \frac{\partial \phi_2(y)}{\partial A_a(z)},
\end{aligned} \tag{410}
$$

but from the information above, we only really need to look at

$$\frac{\partial \phi_2(y)}{\partial A_0(z)} = -2A_0(y)\left((A_0(y))^2 - (A_i(y))^2 \pm (b)^2\right). \tag{411}$$

Thus we see that, finally,

$$\begin{aligned}
&\left[\Pi^0(x), \int dy\ v_2(y)\phi_2(y)\right] = \int dy\ v_2(y)\left[\Pi^0(x), \phi_2(y)\right] \\
&= -\int dz \int dy\ v_2(y)\frac{\partial \Pi^0(x)}{\partial \Pi^a(z)}\frac{\partial \phi_2(y)}{\partial A_a(z)} \\
&= -\int dz \int dy\ v_2(y)(\delta^0_a \delta(z-x))(-2A_0(y)\left((A_0(y))^2 - (A_i(y))^2 \pm (b)^2\right)) \\
&= 2A_0(x)v_2(x)\left((A_0(x))^2 - (A_i(x))^2 \pm (b)^2\right).
\end{aligned} \tag{412}$$

From the information above, we have effectively shown that

$$\begin{aligned}
\boldsymbol{\dot{\Pi}^0} &\boldsymbol{= 2\left(\frac{(a_1)^2 - (a_3)^2}{a_1}\right)\partial_i\partial_i A_0(x) - \left(\frac{a_3}{a_1}\right)\partial_i \Pi^i(x)} \\
&\boldsymbol{+ (2A_0(x)v_2(x) - 2\lambda A_0(x))\left((A_0(x))^2 - (A_i(x))^2 \pm (b)^2\right).}
\end{aligned} \tag{413}$$

Now we turn our attention to

$$\begin{aligned}
\dot{\Pi}^i &= \left[\Pi^i(x), H_E\right] \\
&= \left[\Pi^i(x), H\right] + \left[\Pi^i(x), \int dy\ v_1(y)\phi_1(y)\right] + \left[\Pi^i(x), \int dy\ v_2(y)\phi_2(y)\right];
\end{aligned} \tag{414}$$

Again we'll begin with

$$\left[\Pi^i(x), H\right] = -\int dz\ \frac{\partial \Pi^i(x)}{\partial \Pi^a(z)}\frac{\partial H}{\partial A_a(z)}. \tag{415}$$

By definition

$$\frac{\partial \Pi^i(x)}{\partial \Pi^a(z)} = \delta^i_a \delta(z-x), \tag{416}$$

and thus we're only interested in

$$\begin{aligned}
\frac{\partial H}{\partial A_i(z)} &= 2a_1(\partial_k\partial_k A_i(z)) + 2\left(\left(-\frac{(a_2)^2}{(a_1+a_2+a_3)}\right) + (a_2+a_3)\right)(\partial_i\partial_k A_k(x)) \\
&- \left(\frac{a_2}{(a_1+a_2+a_3)}\right)\partial_i \Pi^0(z) - 2\lambda A_i(z)\left((A_0(z))^2 - (A_i(z))^2 \pm (b)^2\right).
\end{aligned} \tag{417}$$

Thus,

$$\begin{aligned}
\left[\Pi^i(x), H\right] &= -2a_1(\partial_k\partial_k A_i(x)) + 2\left(\left(\frac{(a_2)^2}{(a_1+a_2+a_3)}\right) - (a_2+a_3)\right)(\partial_i\partial_k A_k(x)) \\
&+ \left(\frac{a_2}{(a_1+a_2+a_3)}\right)\partial_i \Pi^0(x) + 2\lambda A_i(x)\left((A_0(x))^2 - (A_i(x))^2 \pm (b)^2\right).
\end{aligned} \tag{418}$$

Next we consider

$$\left[\Pi^i(x), \int dy\ u_1(y)\phi_1(y)\right] = \int dy\ u_1(y)\left[\Pi^i(x), \phi_1(y)\right] \\ = -\int dz \int dy\ u_1(y)\frac{\partial \Pi^i(x)}{\partial \Pi^a(z)}\frac{\partial \phi_1(y)}{\partial A_a(z)}. \tag{419}$$

But again, we know that $\frac{\partial \phi_1(y)}{\partial A_a(z)} = 0$, and thus we have that

$$\left[\Pi^i(x), \int dy\ u_1(y)\phi_1(y)\right] = \mathbf{0}. \tag{420}$$

Lastly, we look at

$$\left[\Pi^i(x), \int dy\ v_2(y)\phi_2(y)\right] = \int dy\ v_2(y)\left[\Pi^i(x), \phi_2(y)\right] \\ = -\int dz \int dy\ v_2(y)\frac{\partial \Pi^i(x)}{\partial \Pi^a(z)}\frac{\partial \phi_2(y)}{\partial A_a(z)}, \tag{421}$$

but again from the information above, we're only concerned with

$$\frac{\partial \phi_2(y)}{\partial A_i(z)} = 2A_i(y)\left((A_0(y))^2 - (A_i(y))^2 \pm (b)^2\right), \tag{422}$$

which means that

$$\begin{aligned} &\left[\Pi^i(x), \int dy\ v_2(y)\phi_2(y)\right] = \int dy\ v_2(y)\left[\Pi^i(x), \phi_2(y)\right] \\ &= -\int dz \int dy\ v_2(y)\frac{\partial \Pi^i(x)}{\partial \Pi^a(z)}\frac{\partial \phi_2(y)}{\partial A_a(z)} \\ &= -\int dz \int dy\ v_2(y)(\delta^i_a\delta(z-x))(2A_i(y)\left((A_0(y))^2 - (A_i(y))^2 \pm (b)^2\right)) \\ &= -2A_i(x)v_2(x)\left((A_0(x))^2 - (A_i(x))^2 \pm (b)^2\right). \end{aligned} \tag{423}$$

From the information above, we have effectively shown that

$$\begin{aligned} \boldsymbol{\dot{\Pi}^i =} & \boldsymbol{2\left(\left(\frac{(a_2)^2}{(a_1+a_2+a_3)}\right) - (a_2+a_3)\right)(\partial_i\partial_k A_k(x))} \\ & \boldsymbol{+ \left(\frac{a_2}{(a_1+a_2+a_3)}\right)\partial_i\Pi^0(x) - 2a_1(\partial_k\partial_k A_i(x))} \\ & \boldsymbol{+ (2\lambda A_i(x) - 2A_i(x)v_2(x))\left((A_0(x))^2 - (A_i(x))^2 \pm (b)^2\right).} \end{aligned} \tag{424}$$

And last but not least, we turn our focus to

$$\begin{aligned} \dot{\Pi}^\lambda &= \left[\Pi^\lambda(x), H_E\right] \\ &= \left[\Pi^\lambda(x), H\right] + \left[\Pi^\lambda(x), \int dy\ v_1(y)\phi_1(y)\right] + \left[\Pi^\lambda(x), \int dy\ v_2(y)\phi_2(y)\right] \end{aligned} \tag{425}$$

Starting with

$$\Big[\Pi^{\lambda}(x), H\Big] = -\int dz\ \frac{\partial \Pi^{\lambda}(x)}{\partial \Pi^{a}(z)} \frac{\partial H}{\partial A_a(z)}, \tag{426}$$

we know that

$$\frac{\partial \Pi^{\lambda}(x)}{\partial \Pi^{a}(z)} = \delta^{\lambda}_{a}\delta(z-x) \tag{427}$$

by definition, and thus we're only concerned with

$$\frac{\partial H}{\partial A_{\lambda}(z)} = \frac{1}{2}\left((A_0(z))^2 - (A_i(z))^2 \pm (b)^2\right)^2. \tag{428}$$

It is thus easy to show that

$$\Big[\Pi^{\lambda}(x), H\Big] = -\frac{1}{2}\left((A_0(x))^2 - (A_i(x))^2 \pm (b)^2\right)^2. \tag{429}$$

Now we can look at

$$\begin{aligned}\left[\Pi^{\lambda}(x), \int dy\ v_1(y)\phi_1(y)\right] &= \int dy\ v_1(y) \left[\Pi^{\lambda}(x), \phi_1(y)\right] \\ &= -\int dz \int dy\ v_1(y) \frac{\partial \Pi^{\lambda}(x)}{\partial \Pi^{a}(z)} \frac{\partial \phi_1(y)}{\partial A_a(z)}.\end{aligned} \tag{430}$$

But, just as in the two previous momenta derivations, we know that $\frac{\partial \phi_1(y)}{\partial A_a(z)} = 0$, and thus we have again that

$$\left[\Pi^{\lambda}(x), \int dy\ v_1(y)\phi_1(y)\right] = \mathbf{0}. \tag{431}$$

Thus we are only left to consider

$$\begin{aligned}\left[\Pi^{\lambda}(x), \int dy\ v_2(y)\phi_2(y)\right] &= \int dy\ v_2(y) \left[\Pi^{\lambda}(x), \phi_2(y)\right] \\ &= -\int dz \int dy\ v_2(y) \frac{\partial \Pi^{\lambda}(x)}{\partial \Pi^{a}(z)} \frac{\partial \phi_2(y)}{\partial A_a(z)},\end{aligned} \tag{432}$$

But it readily evident that $\frac{\partial \phi_2(y)}{\partial A_a(z)} = 0$, and thus

$$\left[\Pi^{\lambda}(x), \int dy\ v_2(y)\phi_2(y)\right] = \mathbf{0}. \tag{433}$$

And thus, pulling all of this together, we have the final momentum equation of motion:

$$\dot{\boldsymbol{\Pi}}^{\lambda} = -\frac{\mathbf{1}}{\mathbf{2}}\left((\mathbf{A_0}(\mathbf{x}))^{\mathbf{2}} - (\mathbf{A_i}(\mathbf{x}))^{\mathbf{2}} \pm (\mathbf{b})^{\mathbf{2}}\right)^{\mathbf{2}}. \tag{434}$$

Now, from the classifications of the constraints in (366), we know that, according to our *Classical* approach, this vector theory has

$$\mathcal{N} = 10\ Field\ Degrees\ of\ Freedom \tag{435}$$

$$n_1 = 2\ First - Class\ Constraints \tag{436}$$

$$n_2 = 0\ Second - Class\ Constraints \tag{437}$$

Which means that it still has

$$\mathcal{N} - 2n_1 - n_2 = 10 - 2(2) - 0 = \;\mathbf{6}\; unaccounted\,degrees\,of\,freedom. \tag{438}$$

As mentioned before, it can be shown that the number of degrees of freedom above, as determined by the Hamiltonian Constraint Analysis procedure, match those determined by a Lagrangian approach to this system. Also, we note that the number of degrees of freedom for this bumblebee model with the *quadratic* Lagrange-Multiplier potential is the same as that of the model with the *linear* Lagrange-Multiplier potential, despite completely different constraint structures for each of the theories. This is perhaps due to the substitution of the linear, regular approximation χ_2 for the nonlinear constraint ϕ_2 in (349), which models the constraint (83) in the linear case, although the evolution of each system from this point is completely different. Again, it is important to note that this correspondence with the results of the Lagrangian approach rests entirely on our assumption that (31) does hold; these results are only obtained by handling the constraints in a chosen, *particular* way, since it is an example of an *irregular* system as discussed in **Section 4.3**. Attempting to account for the six physical degrees of freedom in the model, we can ascribe four of them to the massless Nambu-Goldstone modes which behave similarly to the two transverse modes of the photon each with corresponding conjugate momenta. However, the remaining two degrees of freedom are left undetermined–they could potentially prove to be a massive mode, or a propagating ghost mode, depending on choices of a_1, a_2, a_3–and require further investigation. In any case, it is evident that this bumblebee theory does not reduce precisely to Electromagnetism as explored in **Section 5.1**, but similarities between the two theories do exist despite the presence of an extra mode here–yet unaccounted for–with direct dependence on initial conditions.

6 Summary & Conclusions

We used Dirac's Hamiltonian Constraint Analysis in flat spacetime to explore the properties of the most general class of vector field theories that exhibit Spontaneous Lorentz Symmetry Breaking, known as bumblebee models. Specifically, we considered three of these models, each endowed with a different potential function, and compared them to Electromagnetism in terms of their constraint structures and the number of physical degrees of freedom present in the theory as determined by the Hamiltonian Constraint Analysis procedure. It is perhaps most convenient to summarize the results of this analysis concisely in tabular form, as shown below:

Table 1: Summary of constraints. Shown for each model are the number of fields, the number of primary (1^o), secondary (2^o), first-class (FC), and second-class (SC) constraints, and the resulting number of independent degrees of freedom (DF) according to Dirac's counting argument.

Theory	**Kinetic Term**	**Potential V**	**Fields**	**1^o**	**2^o**	**FC**	**SC**	**DF**
E&M	$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$	(*None*)	8	1	1	2	0	4
Bumblebee	non-Maxwell	$\lambda(A_\mu A^\mu \pm b^2)$	10	1	3	0	4	6
	(arbitrary a_1, a_2, a_3)	$\frac{1}{2}\kappa(A_\mu A^\mu \pm b^2)^2$	8	0	0	0	0	8
		$\frac{1}{2}\lambda(A_\mu A^\mu \pm b^2)^2$	10	1	1	2	0	6

In examining the results of the table above, it is important to notice that no two theories under consideration possess the same constraint structure. The bumblebee model with the linear Lagrange-Multiplier potential has one primary and three secondary constraints, all of which are second-class, resulting in six unaccounted degrees of freedom. Four of these are ascribed to the massless Nambu-Goldstone modes that behave similar to the two transverse modes of the photon in E&M, each with conjugate momentum. Applying the constraints in the theory to effectively 'constrain away' the other fields and momenta, we find that the remaining mode is associated specifically with the components A_j and Π^j, and could potentially prove to be massive, or a freely propagating ghost mode. The results of the bumblebee model with the 'linearized approximation' to the Lagrange-Multiplier potential are suppressed in the table above, as this theory possesses the same constraint structure as its non-linearized counterpart. The difference between these cases is that in the linearized approximation, the constraints directly dictate that the extra mode is related to A_j and Π^j without any leg work; again we posit that the extra mode could potentially be massive. In the case of the smooth quadratic potential, the model is entirely unconstrained, and all eight field and momenta components are free to propagate, which could prove problematic since it may allow for the introduction of ghost modes into the theory. The theory with the quadratic Lagrange-Multiplier potential proved interesting due to the presence of a nonlinear irregular constraint in the course of its analysis. Choosing to replace the irregular constraint with its regular, linear counterpart, the constraint structure and degrees of freedom in the table above correspond directly with the results of a Lagrangian approach to the theory. Again, this correspondence only exists if we handle this model's constraint analysis in a specific way. In such a manner, the theory has one primary and one secondary constraint, both of which are first class; this leaves six unaccounted physical degrees of freedom in the bumblebee model. Similarly to the linear Lagrange-Multiplier case, we relate four of these degrees of freedom to the massless Nambu-Goldstone modes behaving like photon modes with conjugate momenta, and the extra mode is again posited to be potentially a massive mode, or a propagating ghost. Nonetheless, from the table it is evident that none of the three bumblebee models under consideration reduce directly to E&M. In each of the Lorentz-violating vector field theories under consideration there exist two or more additional degrees of freedom in comparison to Electromagnetism. These extra degrees of freedom are important as possible additional propagating modes and in terms of how they could alter the intial-value problem.

Future research should seek to explicitly account for these extra degrees of freedom inherent in each of the three bumblebee models under consideration above; an investigation of these extra degrees of freedom could shed light on how the arbitrary Will-Nordvedt coefficients in the bumblebee models affect the initial-value problem. In this respect, it may be illuminating to consider specific choices of the arbitrary coefficients a_1, a_2, and a_3–*other* than those considered in [26]–for insight into the various behaviors of the constraints and the equations of motion uncovered by the Hamiltonian Constraint Analysis procedure, and also perhaps to examine the adaptability of each of these theories as alternate explanations of the Einstein-Maxwell theory of Electromagnetism as a result of spontaneous Lorentz violation rather than of local U(1) gauge invariance. Also, a more detailed consideration of the nature of irregular constraints and their appropriate treatment may provide some information about the explicit constraint structure and degrees of freedom inherent in the bumblebee model with the quadratic Lagrange-Multiplier discussed above. Investigations of the stability of these theories with respect to the positive-definiteness of their Hamiltonians have already begun, and further research in this area will determine the explicit restrictions on the respective phase-spaces of these models, as well as the physical viability of these theories as alternate explanations of E&M. Additionally, a reproduction of this analysis in *curved* spacetime would be extremely insightful in exploring the implications of Lorentz violation in gravity and cosmology, and in seeking alternative explanations for dark energy and dark matter. In this manner, the bumblebee models could be considered not just as effective field theories incorporating spontaneous Lorentz violation, but potentially as modified theories of gravity as well.

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